Lecture 18

Ford-Fulkerson Method (contd.), Max-Flow Min-Cut Theorem

Source: Introduction to Algorithms, CLRS and Kleinberg & Tardos

Augmenting Flows via Residual Networks

- Find $s \rightsquigarrow t$ path P in the **residual network** G_f and its bottleneck capacity δ.
- For every $(u, v) \in P$:
    - If $(u, v) \in E(G)$, add δ flow to (u, v) in f.
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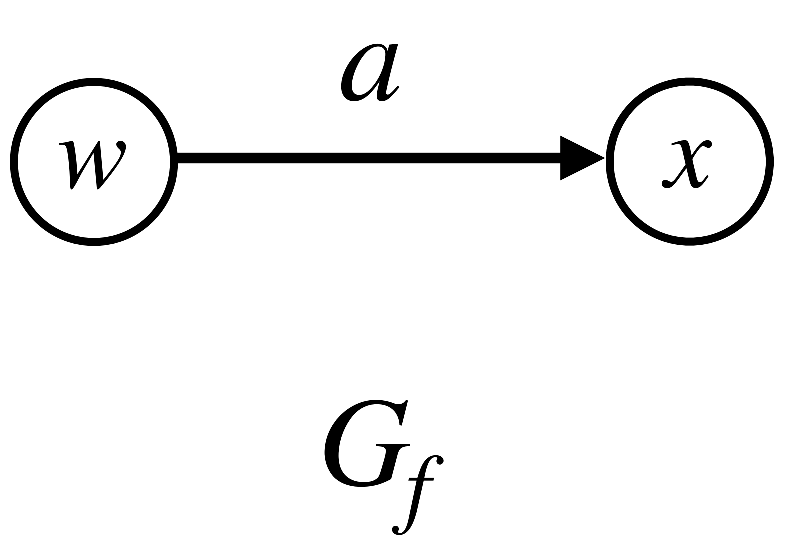

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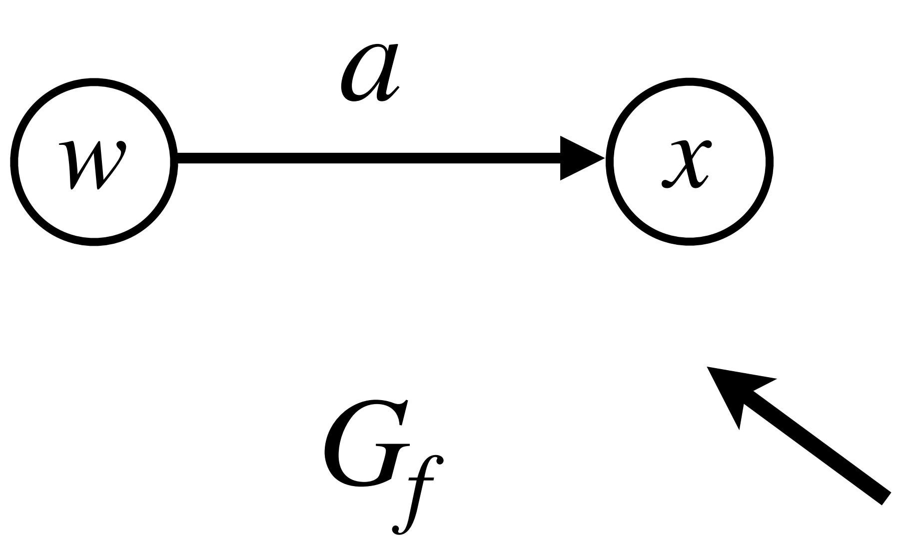

A forward edge
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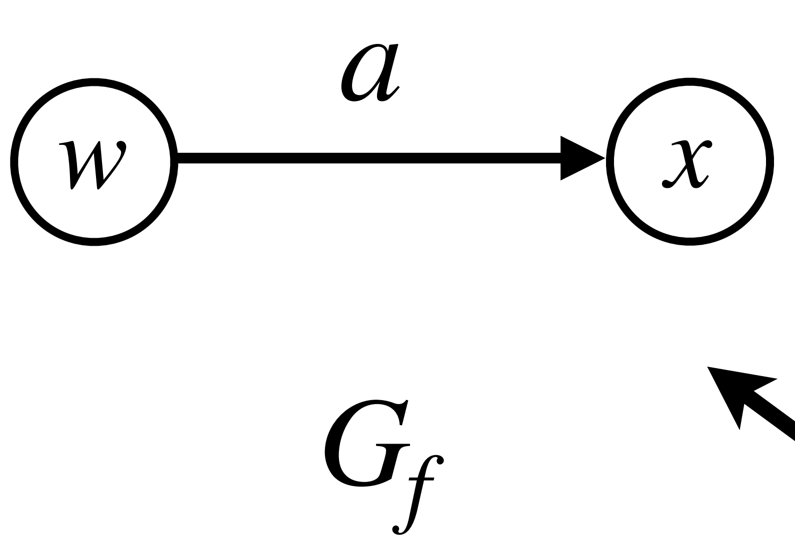

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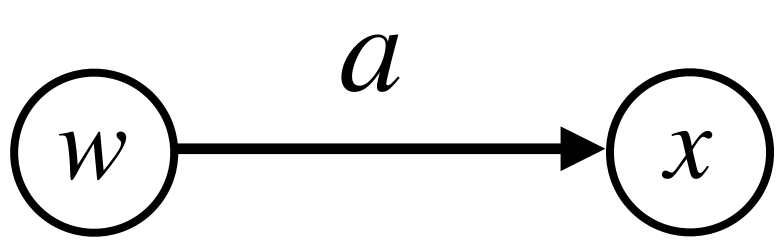

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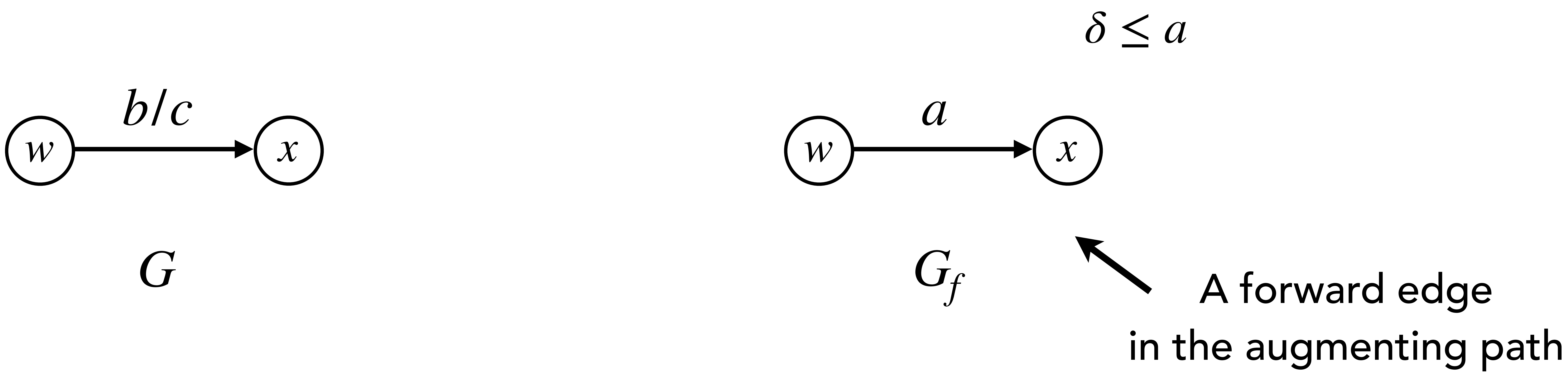

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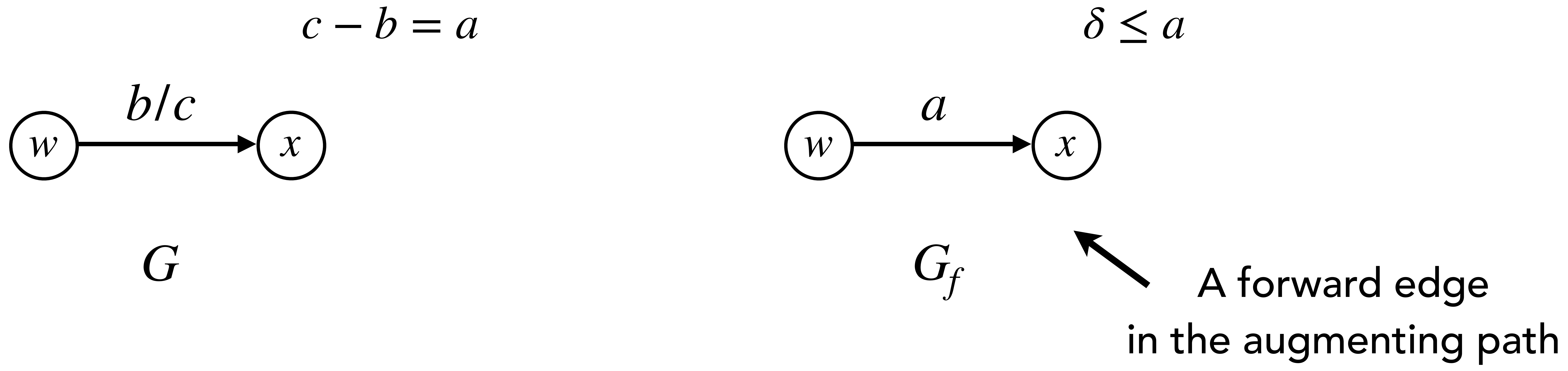

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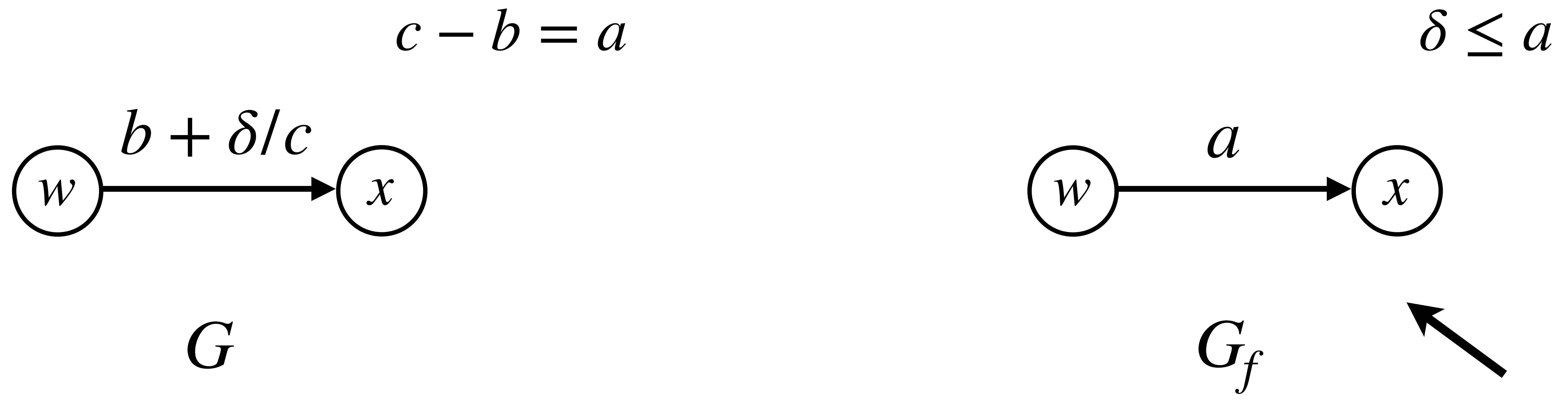

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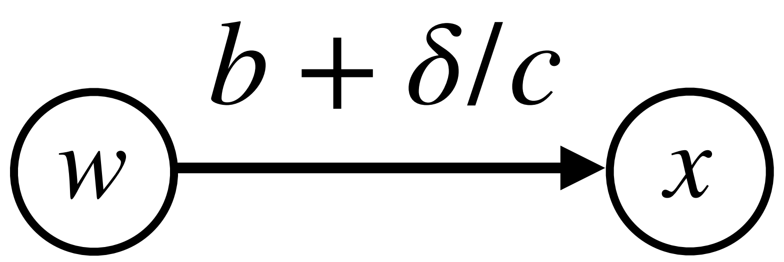
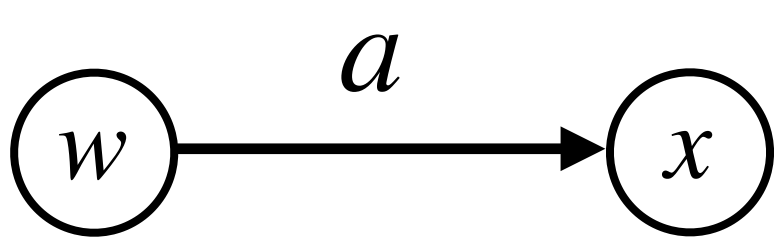
Satisfying Capacity Constraint:

$c - b = a,\ b + \delta \leq c$

$\delta \leq a$



G



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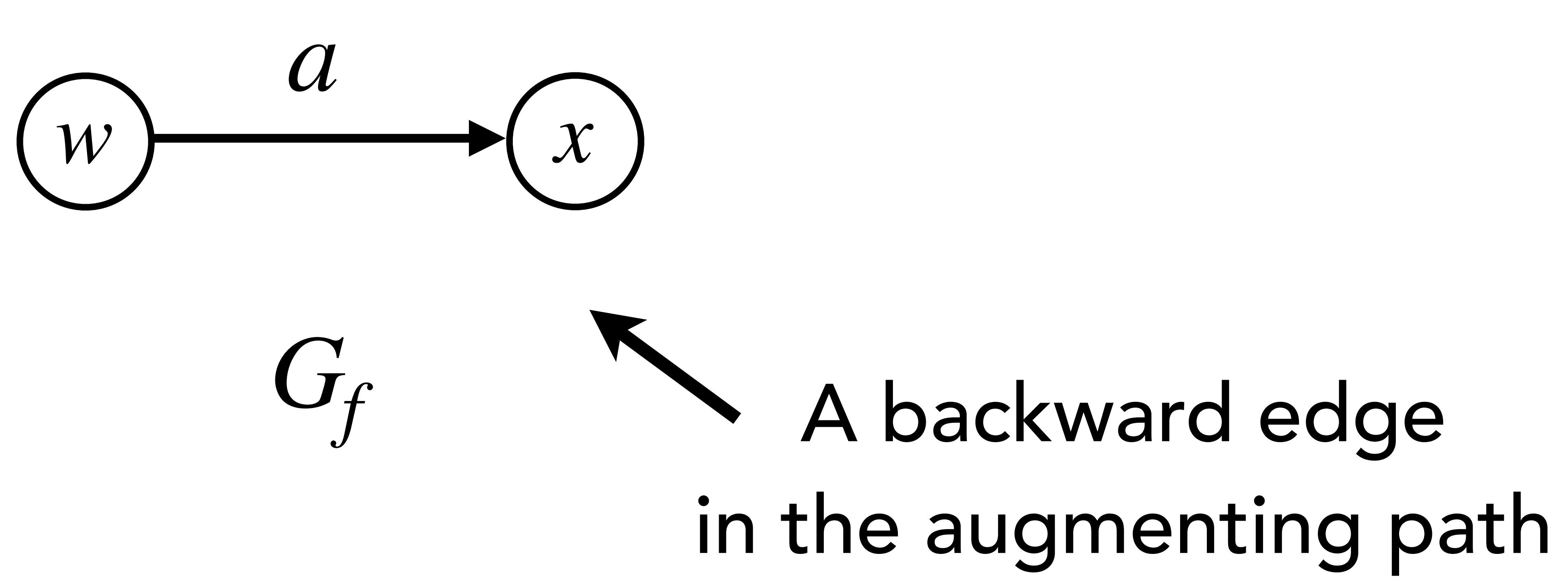

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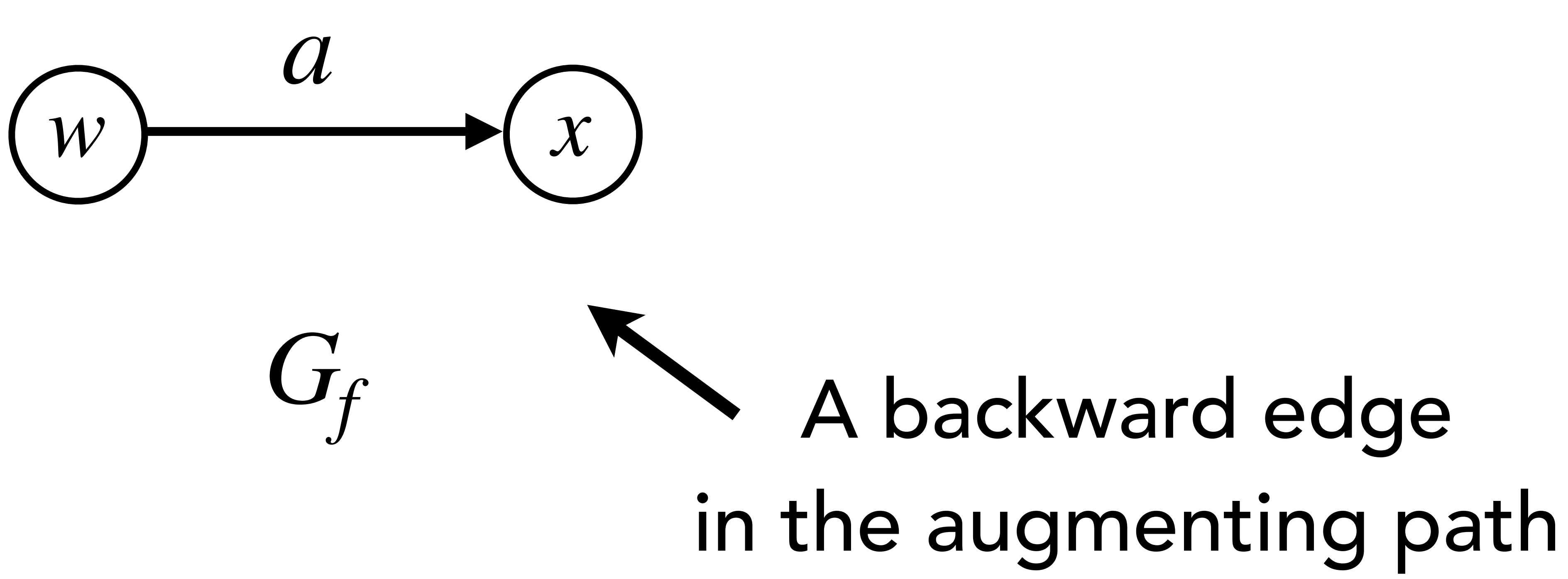

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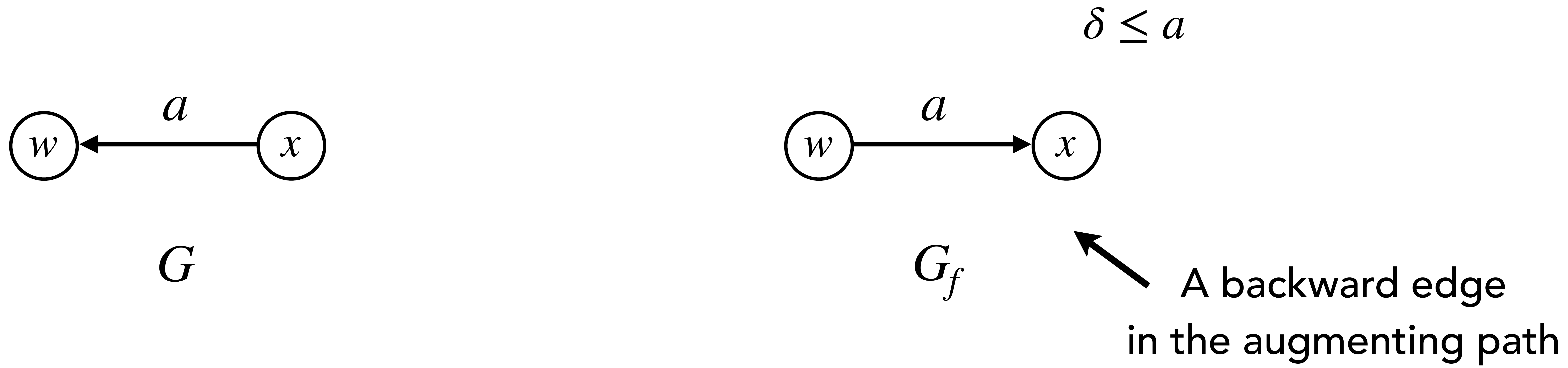

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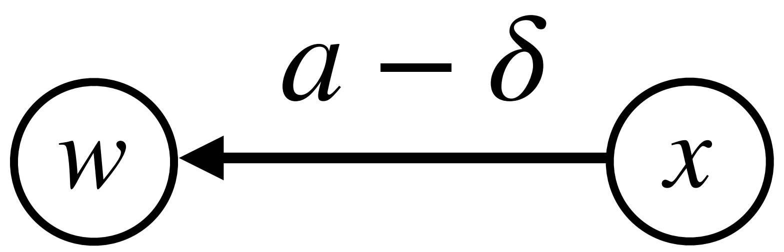

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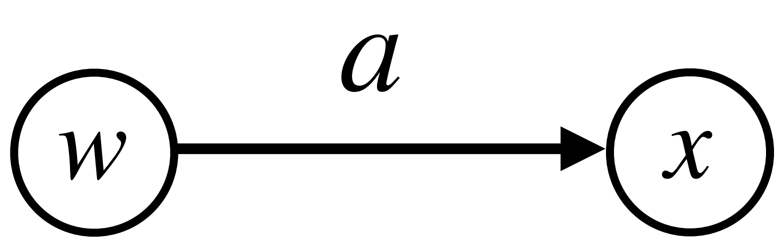

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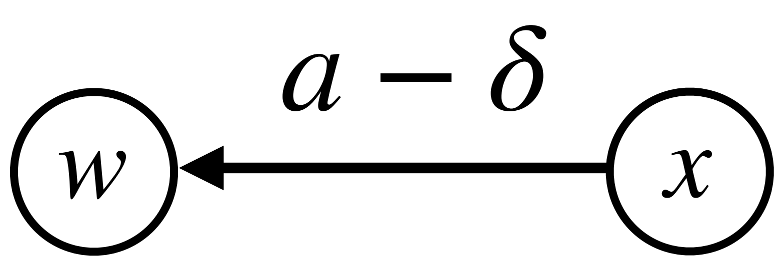
Satisfying Capacity Constraint:

$a - \delta \geq 0$



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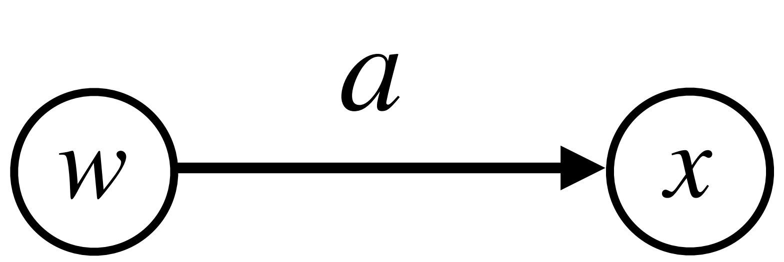

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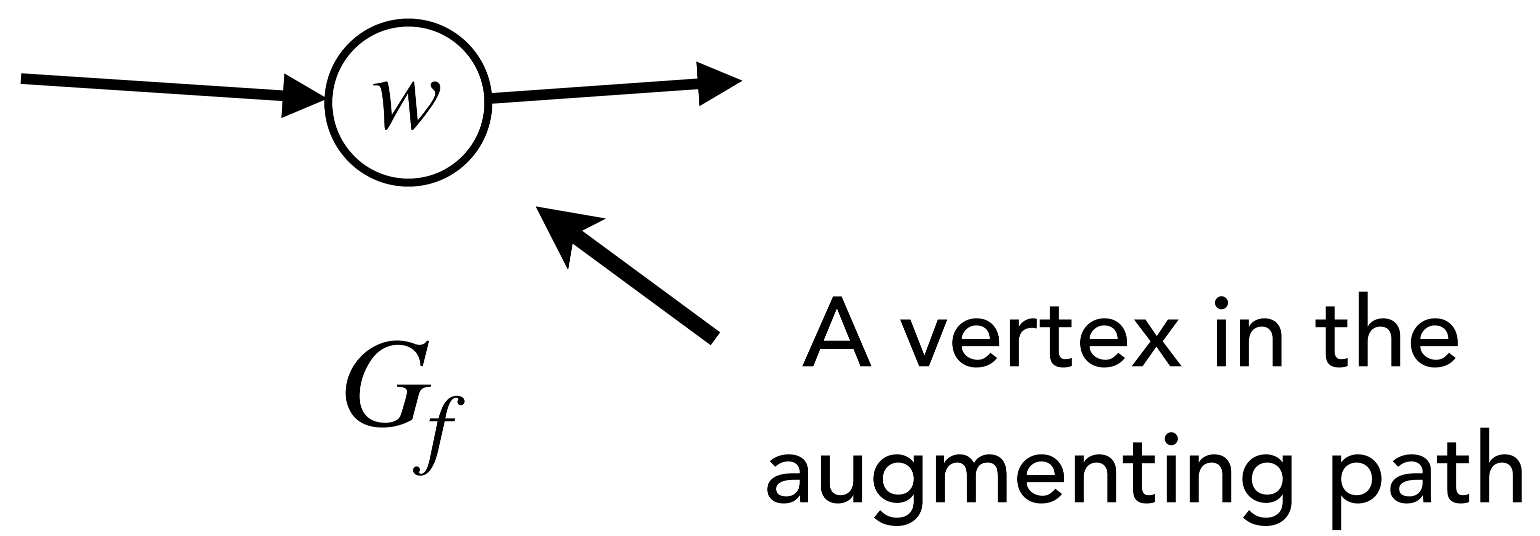

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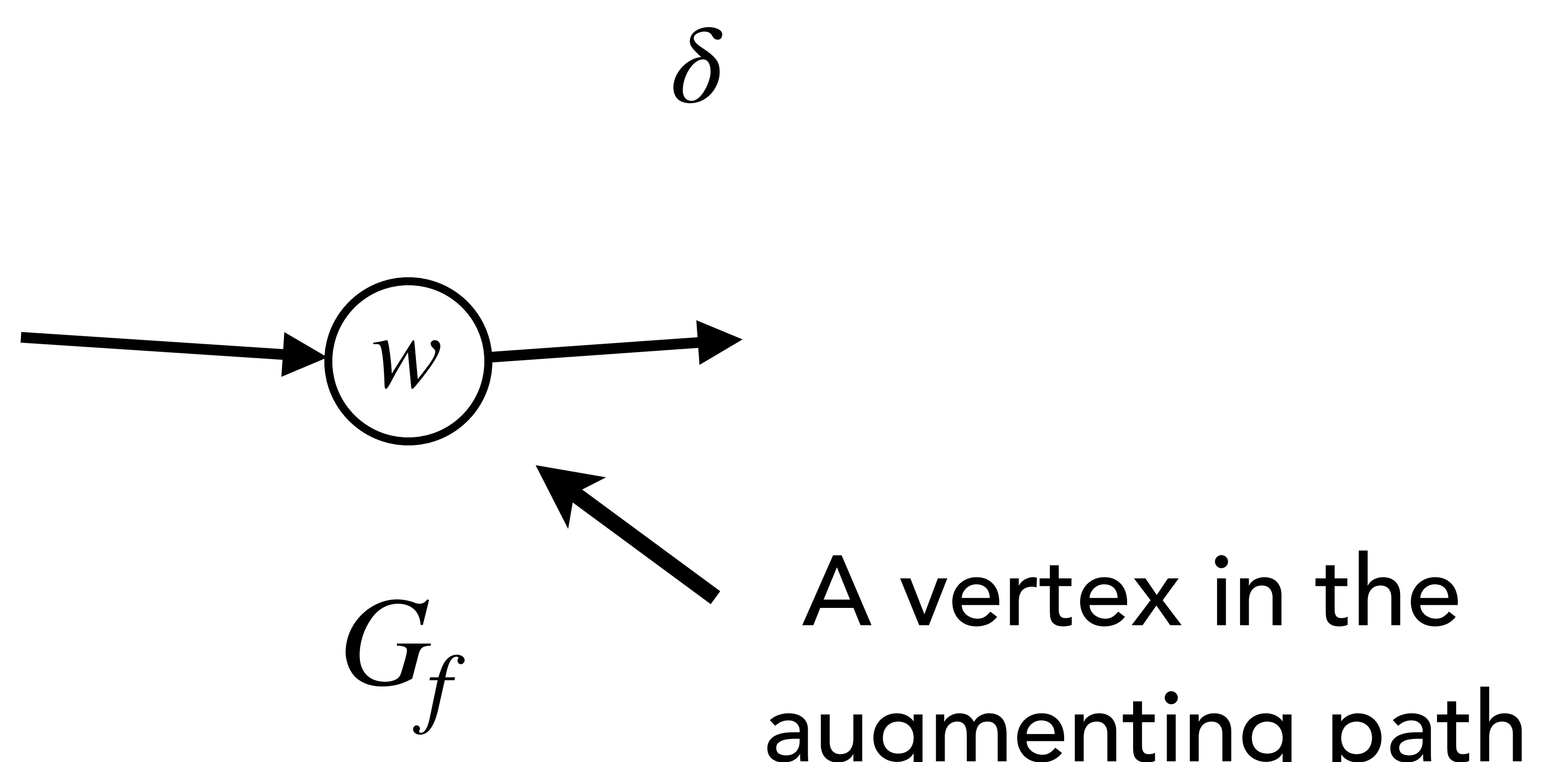

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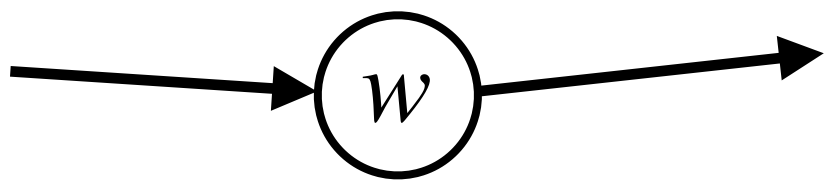
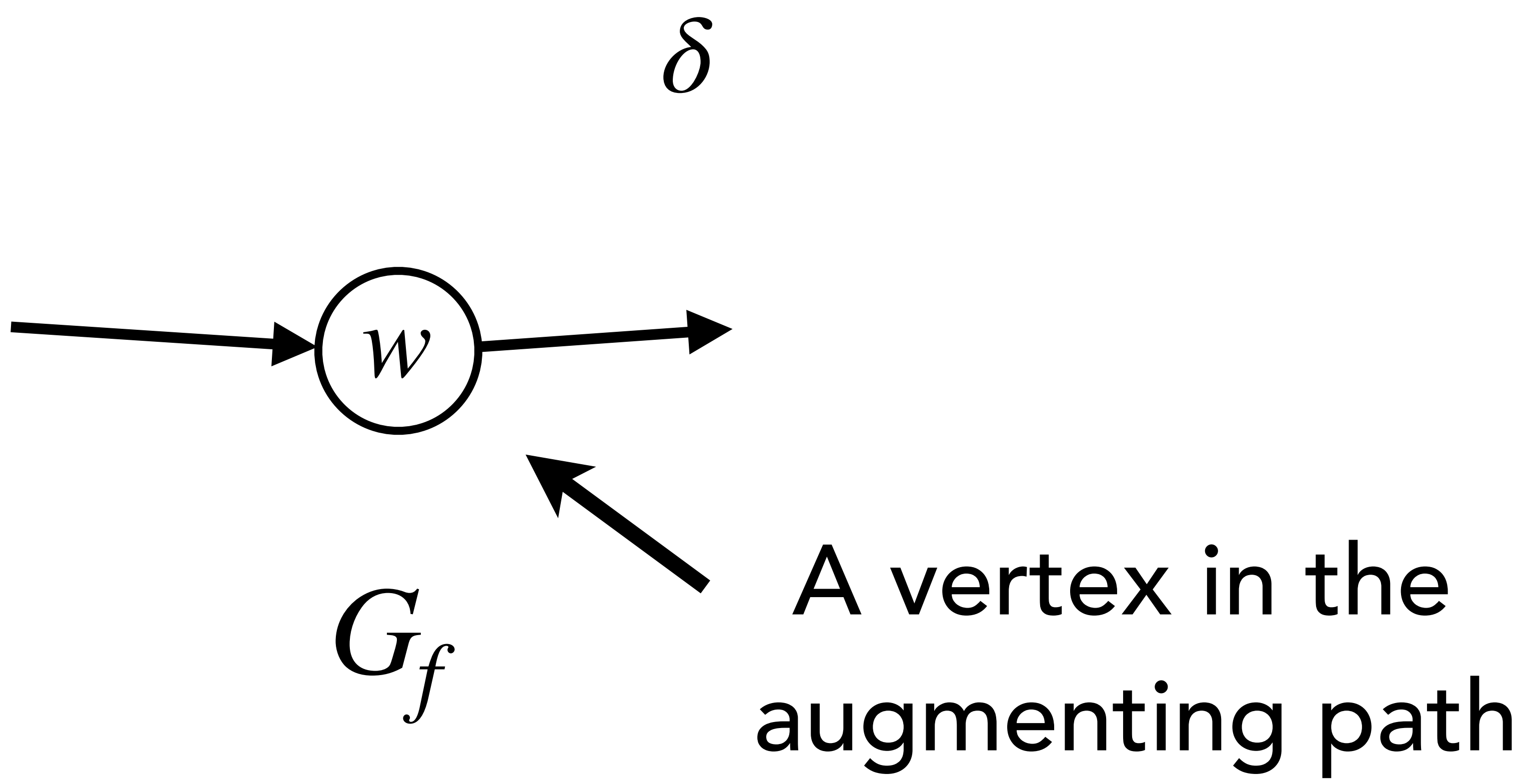


Both edges are forward in G



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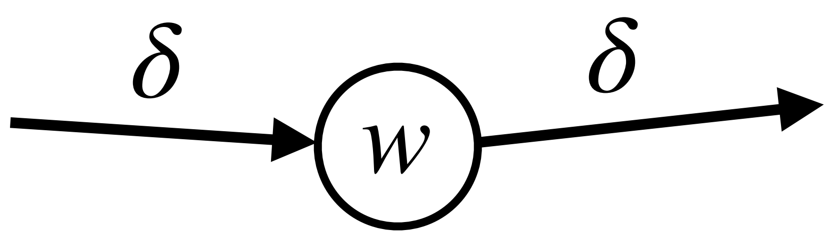

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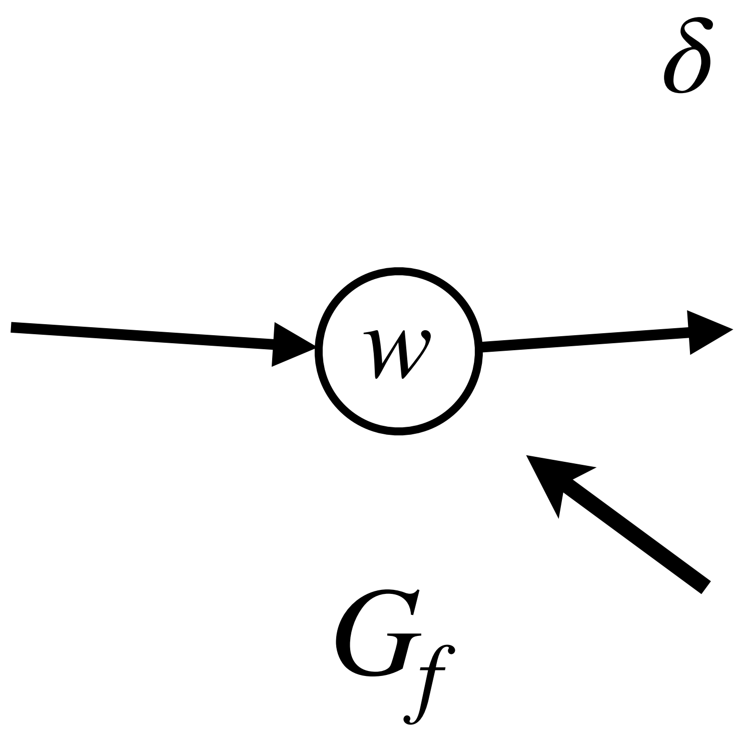

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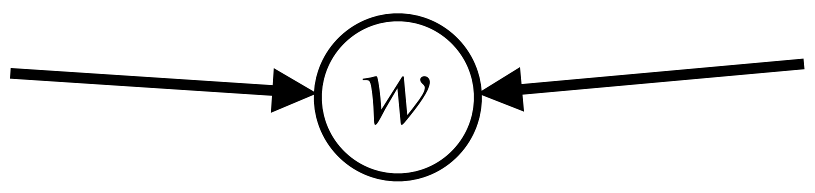
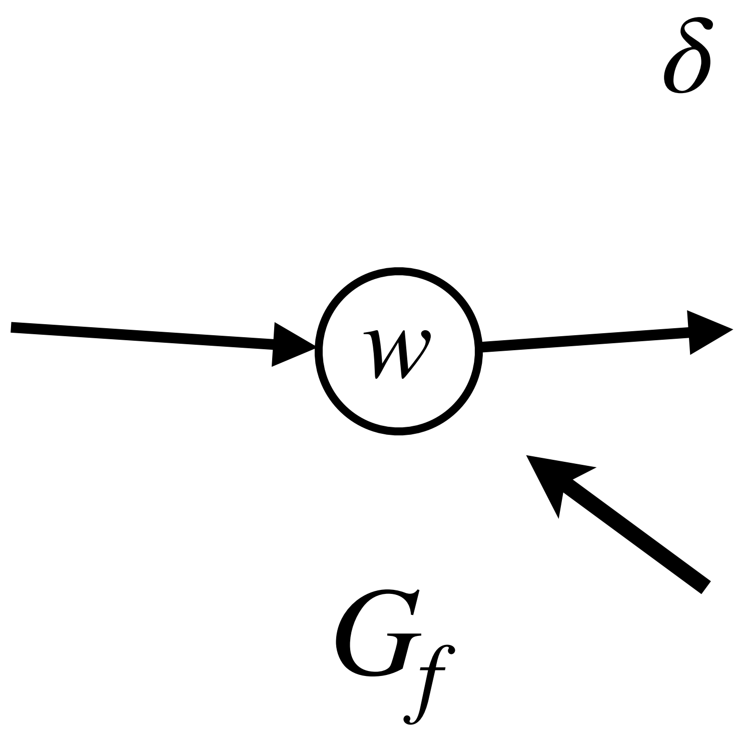


First is forward, second is backward in G



A vertex in the augmenting path

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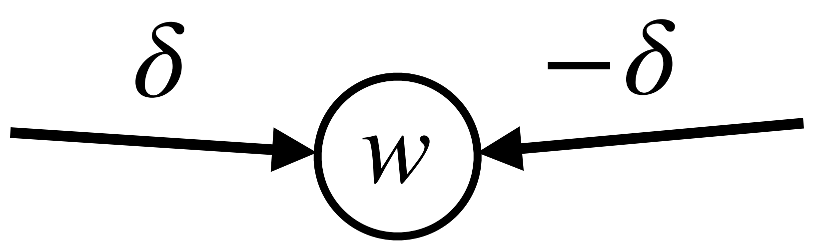

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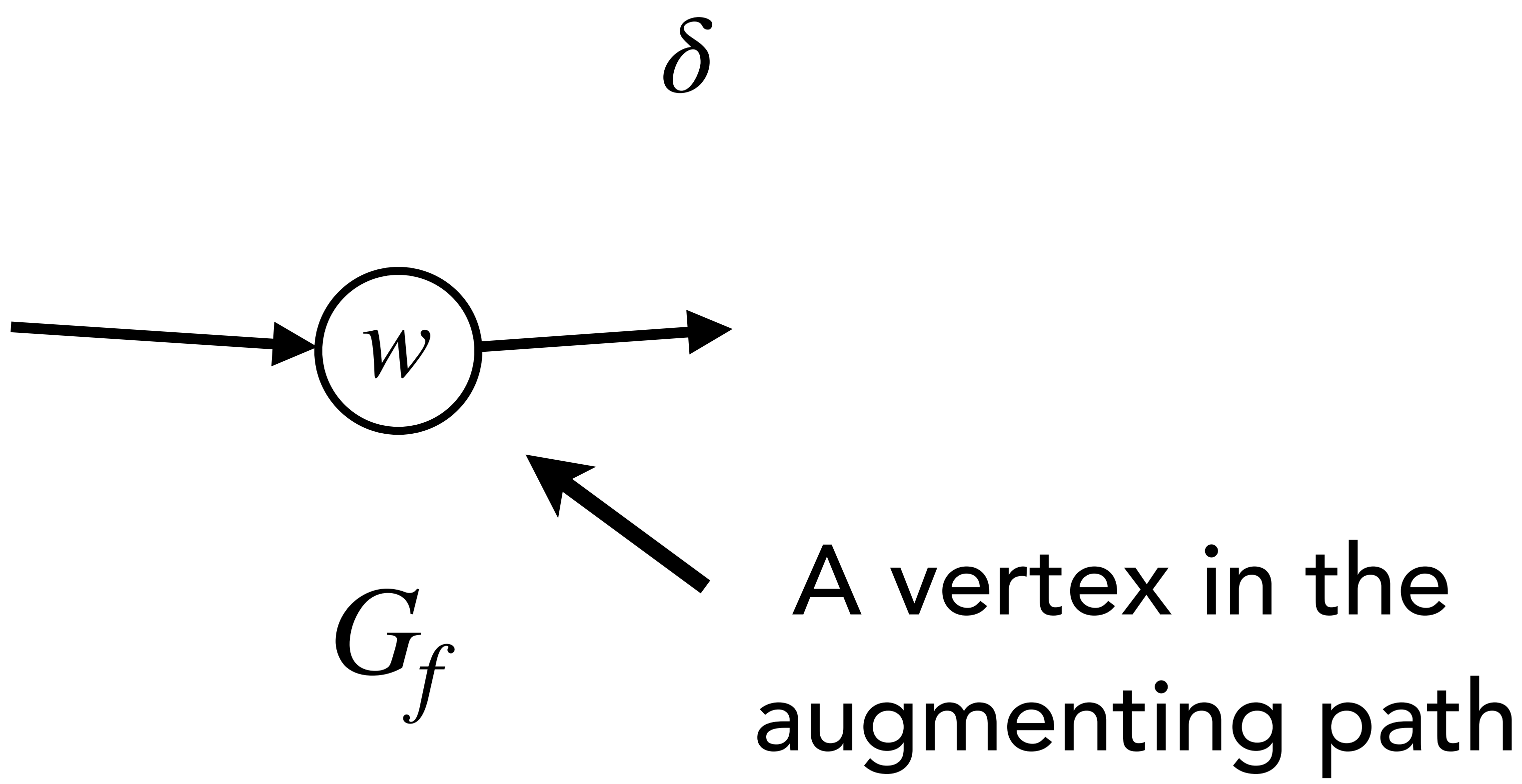

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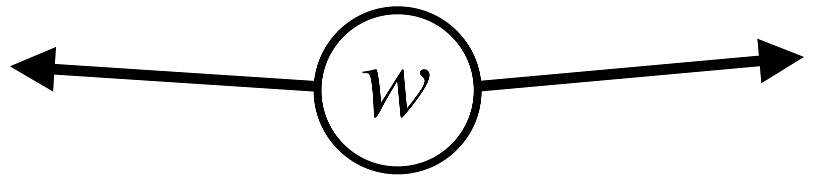

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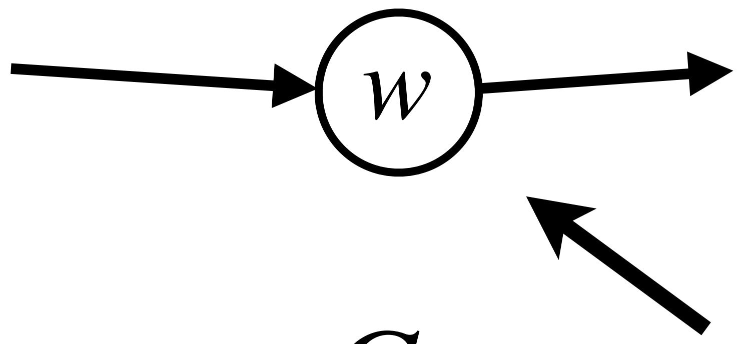
δ



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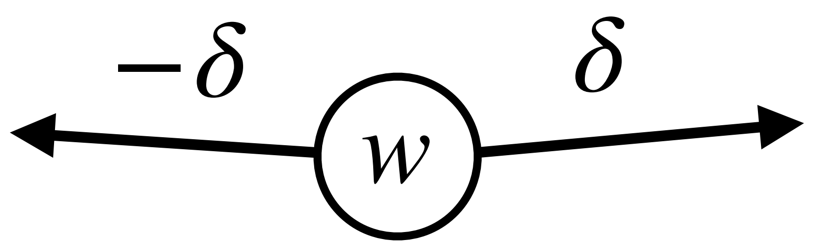

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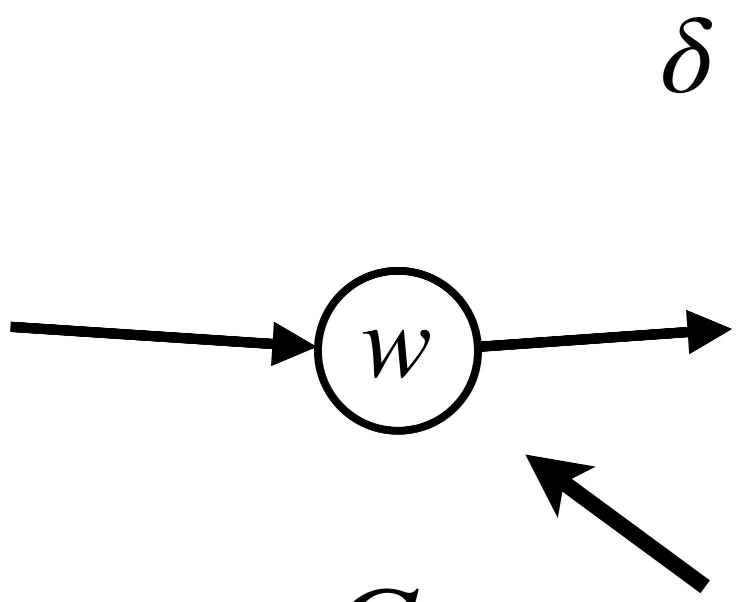

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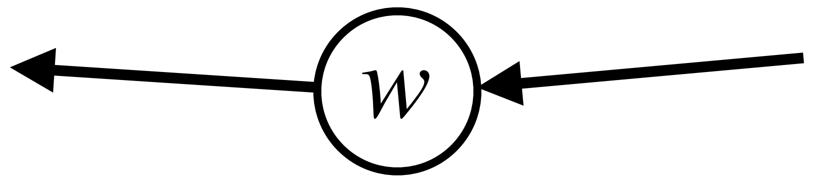

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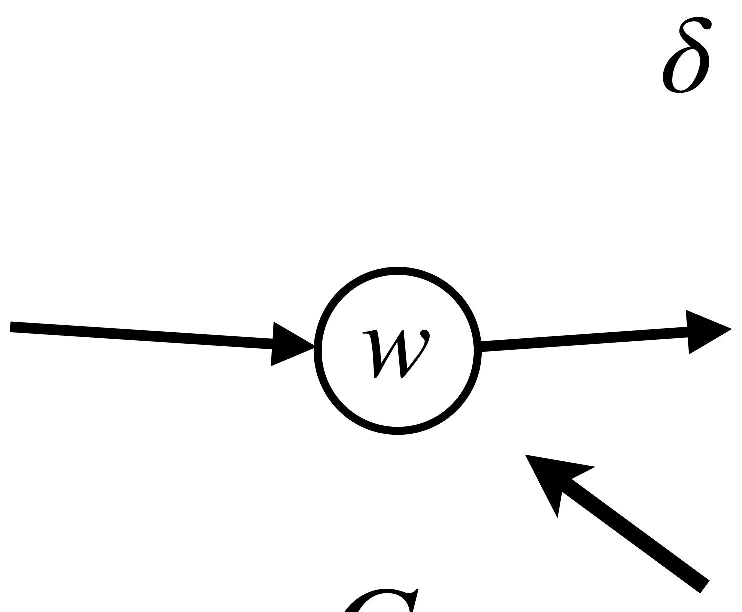

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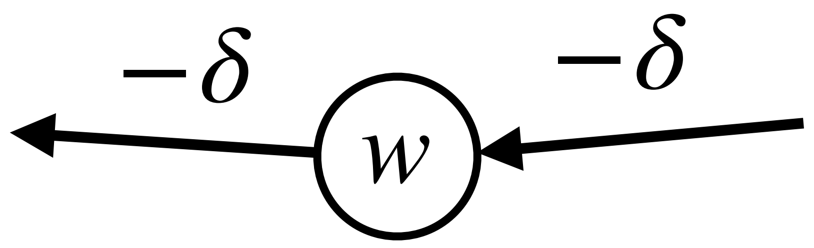

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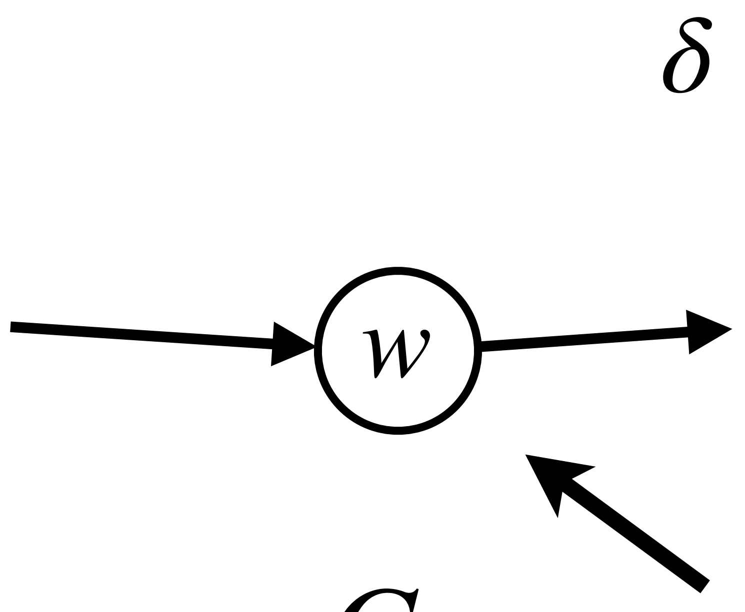

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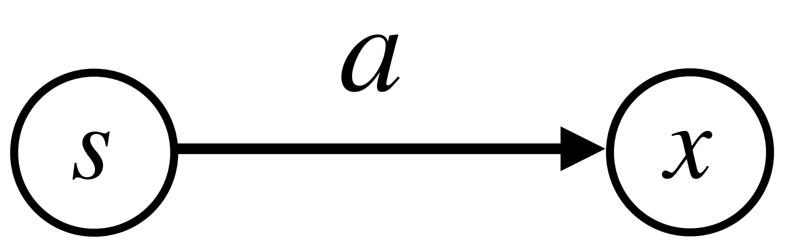

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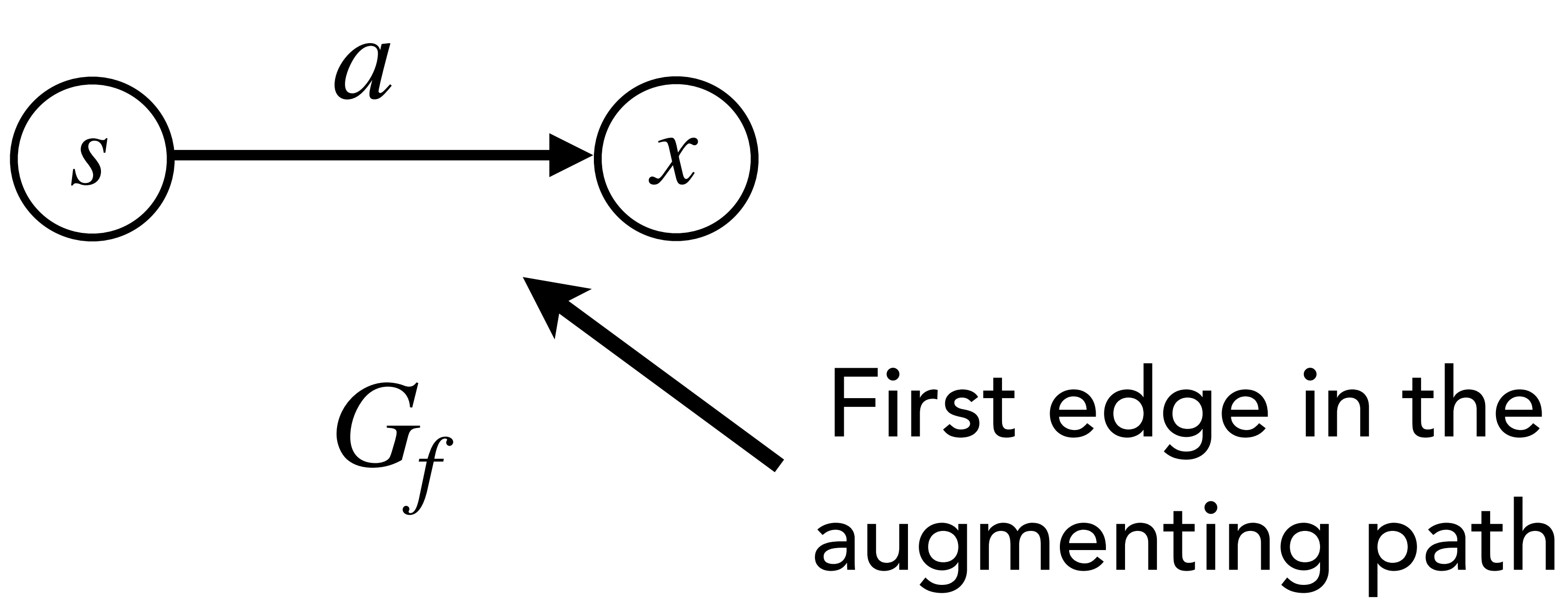
Note: Path P is called an augmenting path.

Why flow value increases?



Augmenting Flows via Residual Networks

- Find $s \rightsquigarrow t$ path P in the **residual network** G_f and its bottleneck capacity δ.
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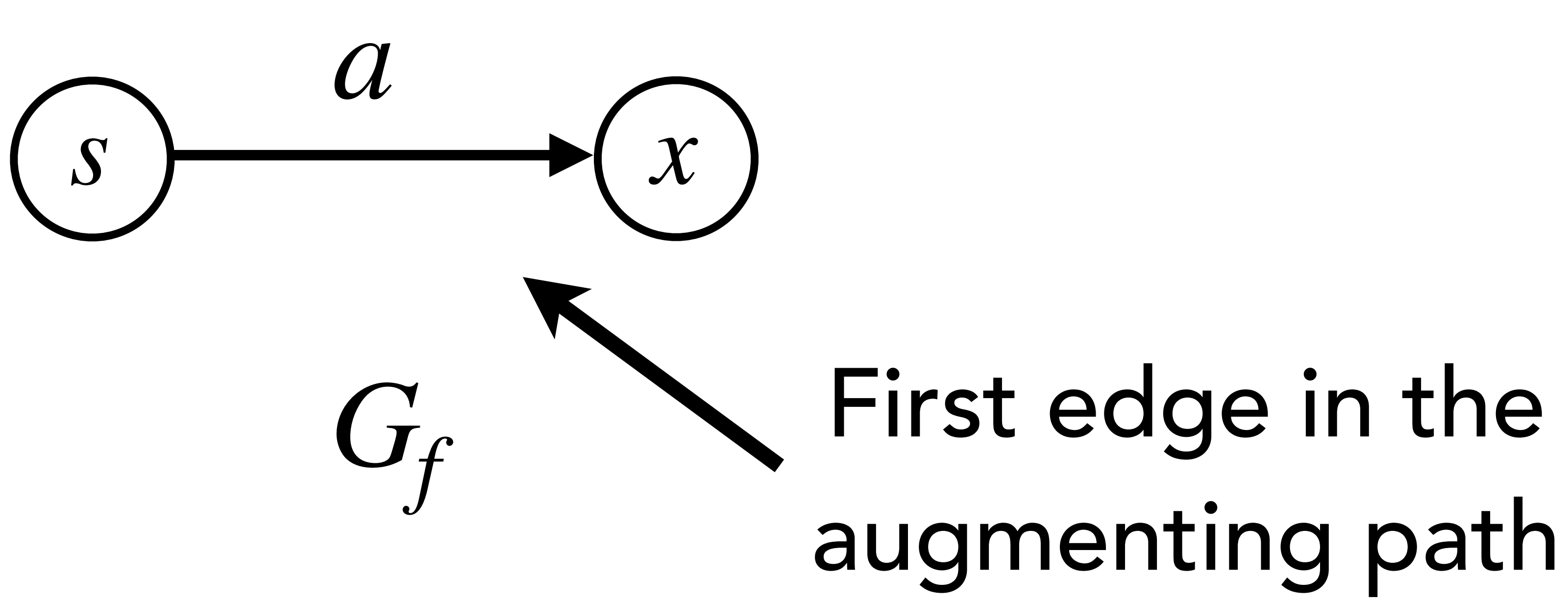
Why flow value increases?

$\delta > 0$



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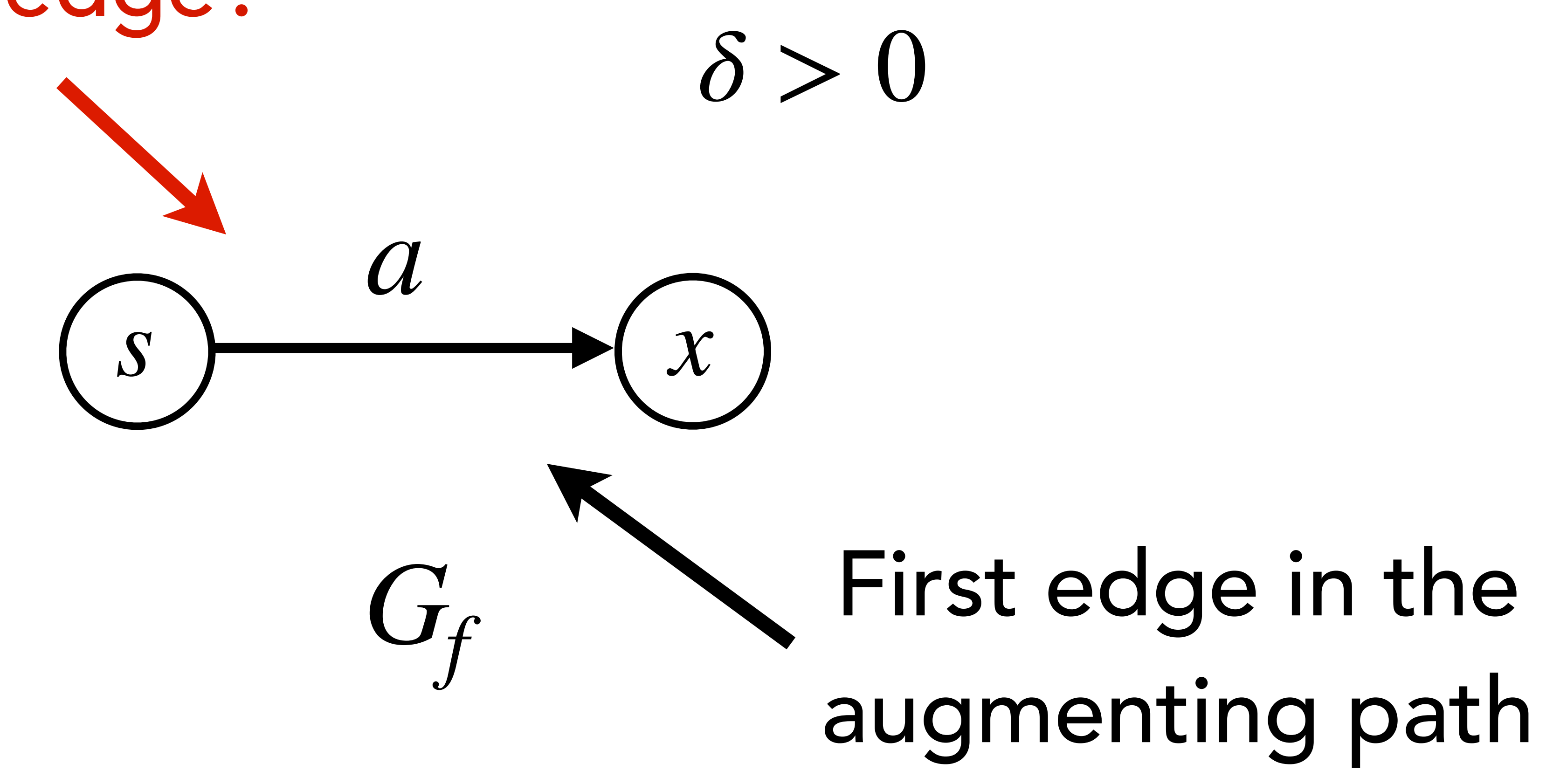
Why flow value increases?

Can it be a backward edge?



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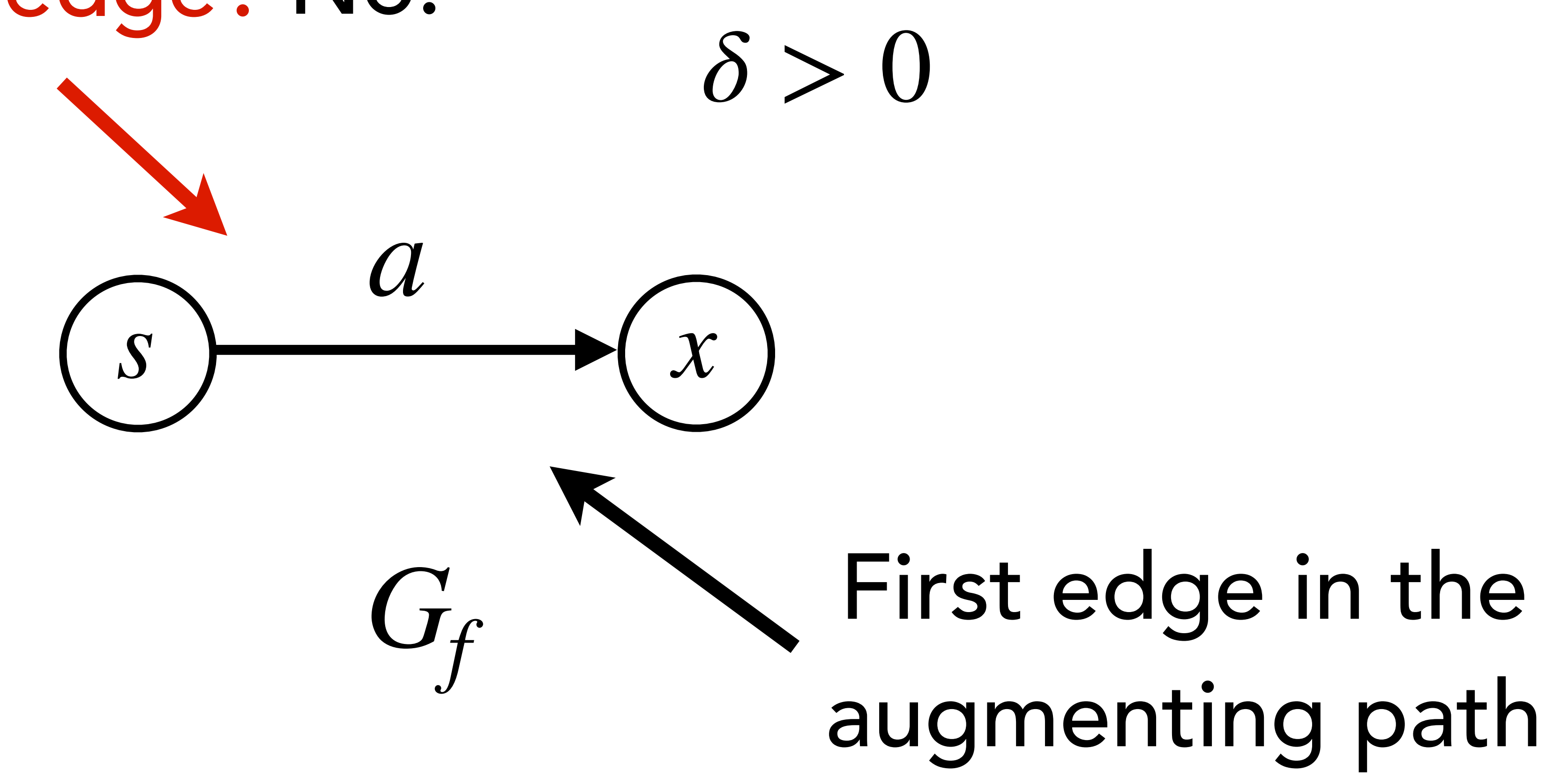

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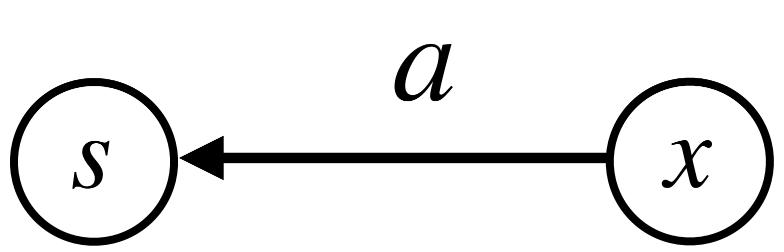
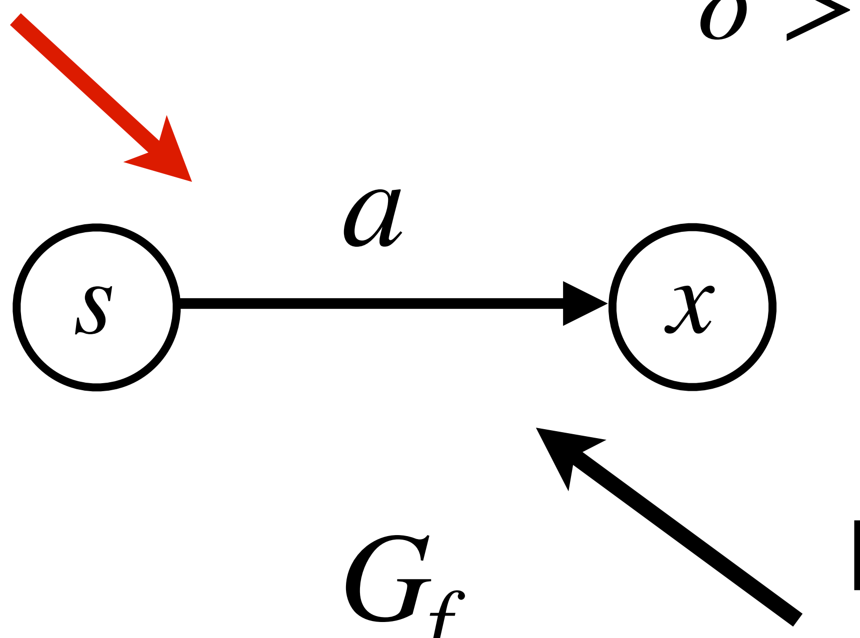


G



G_f

First edge in the augmenting path

Augmenting Flows via Residual Networks

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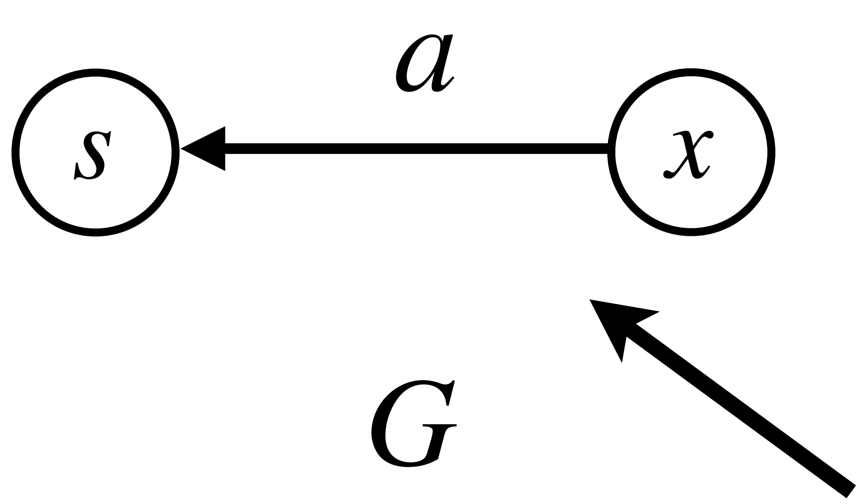
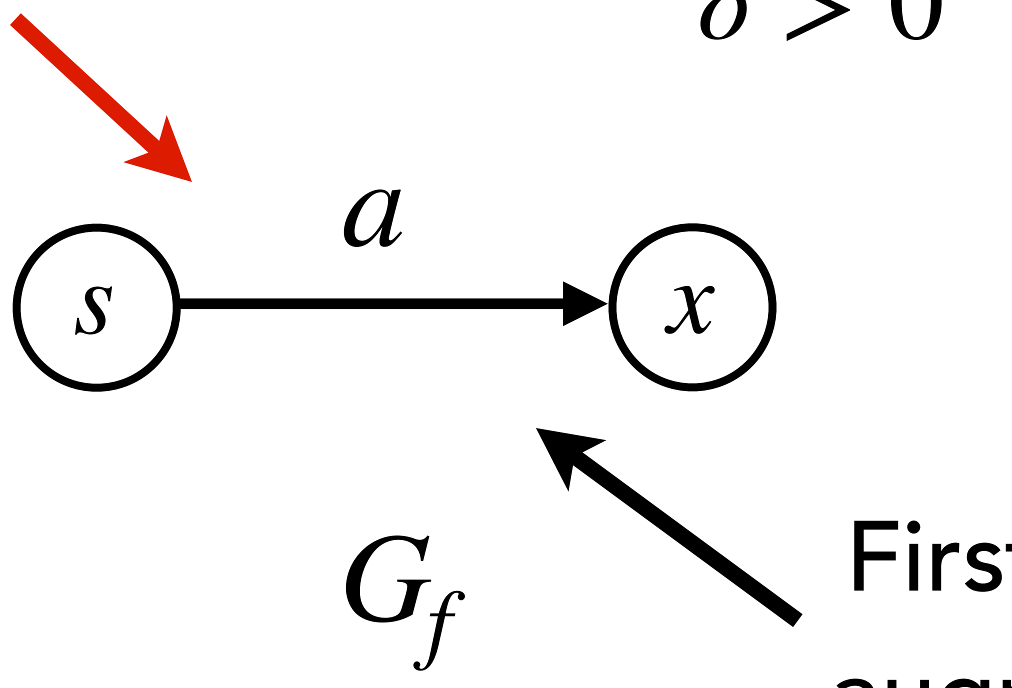


An incoming edge
to s is not possible



First edge in the
augmenting path

Augmenting Flows via Residual Networks

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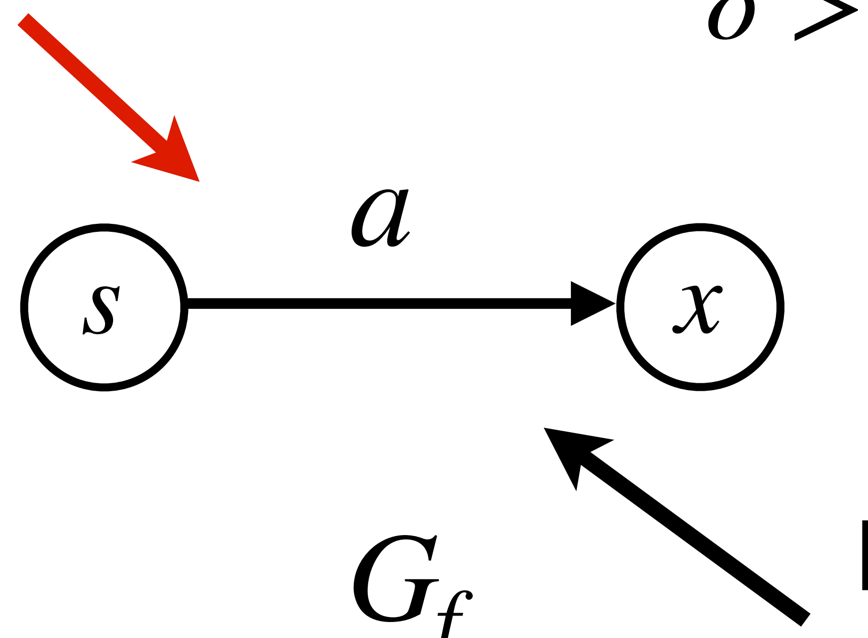
G



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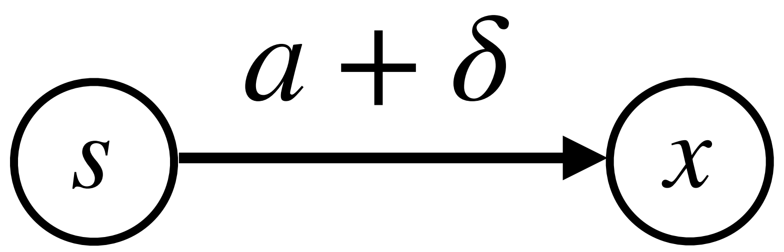

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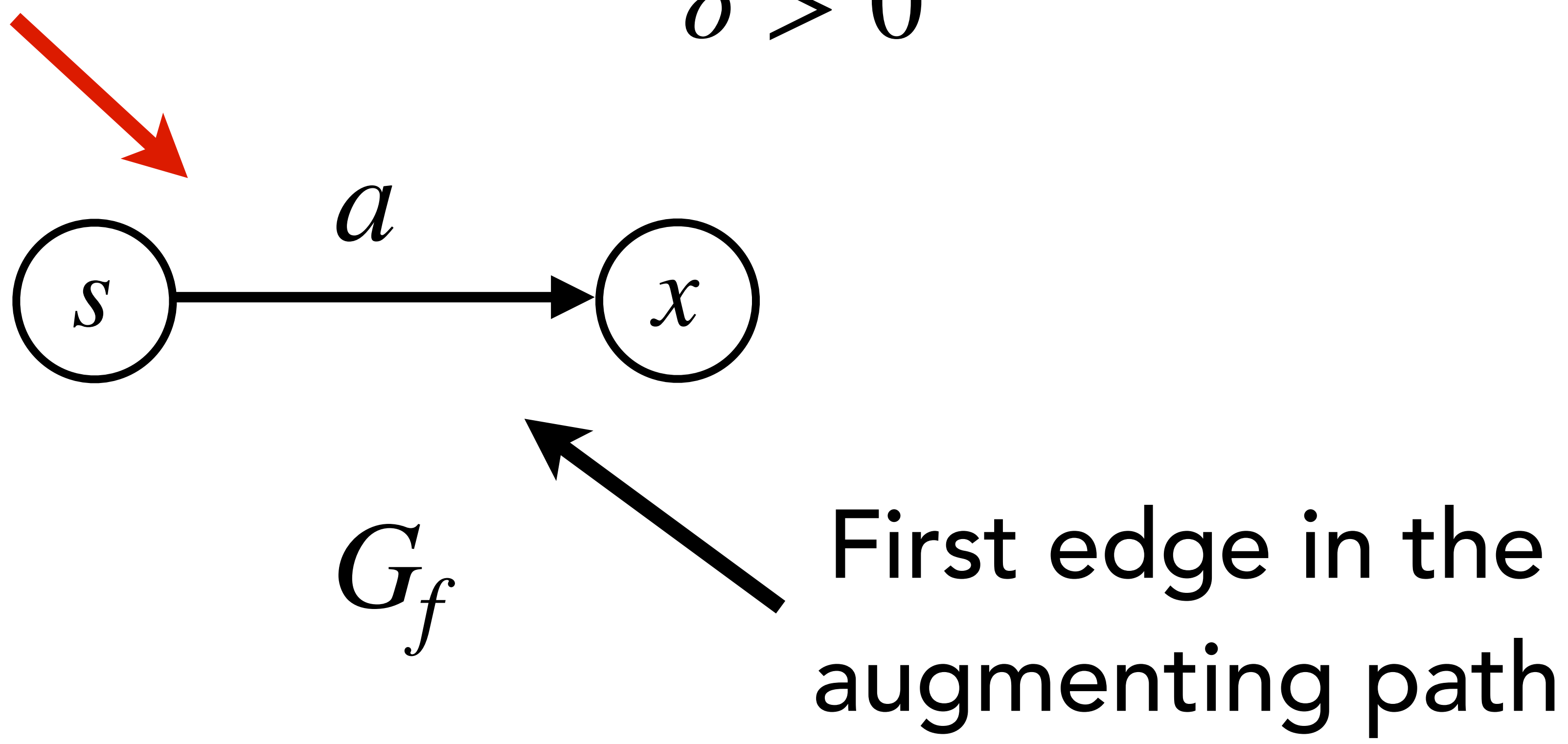

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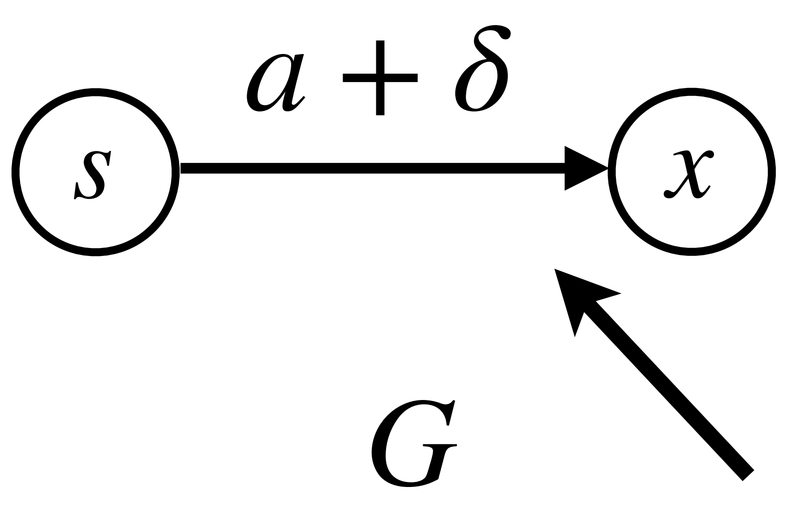

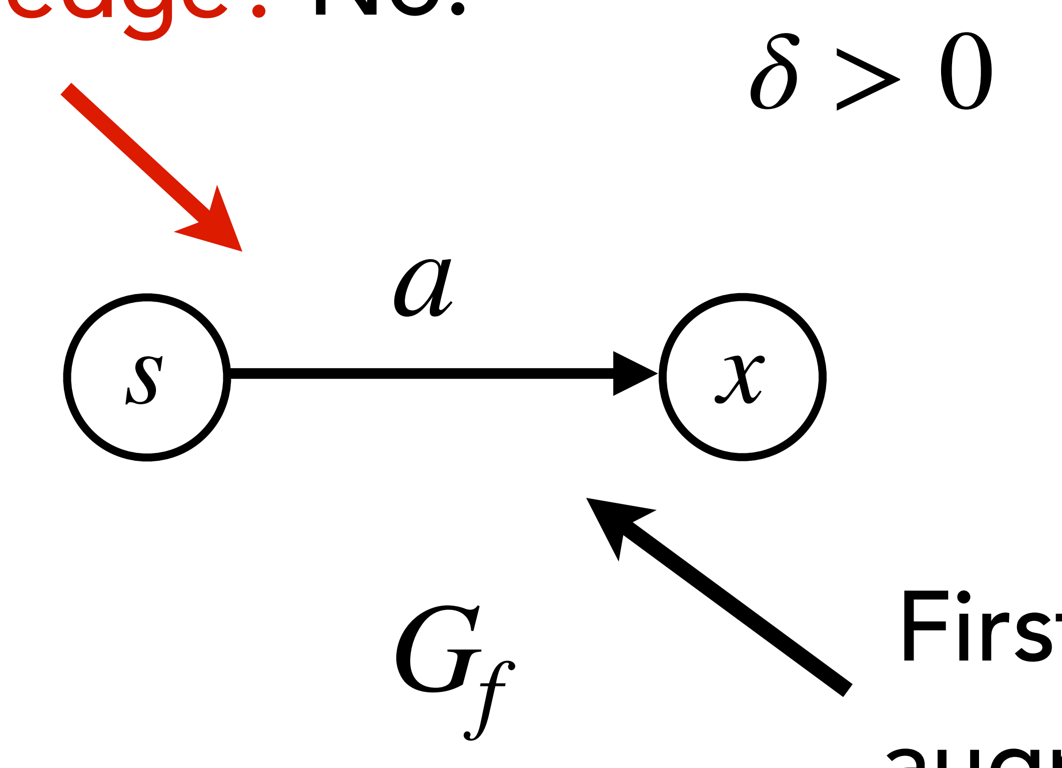

Ford-Fulkerson Method

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Ford-Fulkerson(G, s, t)**:**

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    for each edge (u, v) in P
        if $(u, v) \in E$

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6. $\quad\quad$ **if** $(u, v) \in E$
7. $\quad\quad\quad f(u, v) = f(u, v) + \delta$

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Why f will be maximum when loop breaks?

Ford-Fulkerson Method: Correctness

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Theorem: If there is no augmenting path in the residual network G_f, then f is a maximum flow.

Ford-Fulkerson Method: Correctness

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Proof: We need to study cuts and max-flow, min-cut theorem for the proof.

Ford-Fulkerson Method: Analysis

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Ford-Fulkerson Method: Analysis

Ford-Fulkerson(G, s, t):

for each edge $(u, v) \in E(G)$ ← $O(|E|)$

  $f(u, v) = 0$

while there exists an $s \rightsquigarrow t$ path P in the residual network G_f

  $\delta =$ **Min**$(c_f(u, v) : (u, v)$ **in** $P)$

  for each edge (u, v) in P

    if $(u, v) \in E$

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return f

Ford-Fulkerson Method: Analysis

Ford-Fulkerson(G, s, t)**:**

for each edge $(u, v) \in E(G)$ — $O(|E|)$

  $f(u, v) = 0$

while there exists an $s \rightsquigarrow t$ path P in the residual network G_f — Loop may run for $|f^*|$ time, where f^* is a max-flow, as flow may increase by one with every iteration

  $\delta =$ **Min**$(c_f(u, v) : (u, v)$ **in** $P)$

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Ford-Fulkerson Method: Analysis

```
Ford-Fulkerson(G, s, t):
    for each edge (u, v) ∈ E(G)
        f(u, v) = 0
    while there exists an s ⇝ t path P in the residual network G_f
        δ = Min(c_f(u, v) : (u, v) in P)
        for each edge (u, v) in P
            if (u, v) ∈ E
                f(u, v) = f(u, v) + δ
            else
                f(v, u) = f(v, u) − δ
    return f
```

$O(|E|)$

Loop may run for $|f^*|$ time, where f^* is a max-flow, as flow may increase by one with every iteration

$O(|E|)$ because $|E| \geq |V| - 1$

Ford-Fulkerson Method: Analysis

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- **return** f

$O(|E|)$

Loop may run for $|f^*|$ time, where f^* is a max-flow, as flow may increase by one with every iteration

$O(|E|)$ because $|E| \geq |V| - 1$

Time Complexity: $O(|E| \cdot |f^*|)$

Ford-Fulkerson Method: Analysis

Ford-Fulkerson(G, s, t)**:**

1. **for** each edge $(u, v) \in E(G)$ — $O(|E|)$
2. $\quad f(u, v) = 0$
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4. $\quad \delta =$ **Min**$(c_f(u, v) : (u, v)$ **in** $P)$
5. $\quad$ **for** each edge (u, v) in P
6. $\quad\quad$ **if** $(u, v) \in E$
7. $\quad\quad\quad f(u, v) = f(u, v) + \delta$
8. $\quad\quad$ **else**
9. $\quad\quad\quad f(v, u) = f(v, u) - \delta$
10. **return** f

Loop may run for $|f^*|$ time, where f^* is a max-flow, as flow may increase by one with every iteration

Lines 4–9: $O(|E|)$ because $|E| \geq |V| - 1$

Time Complexity: $O(|E| \cdot |f^*|)$

Note: Analysis is valid when capacities are integer.

Ford-Fulkerson Method: A Non-terminating Case

Ford-Fulkerson Method: A Non-terminating Case

Ford-Fulkerson may not terminate when some capacities are irrational.

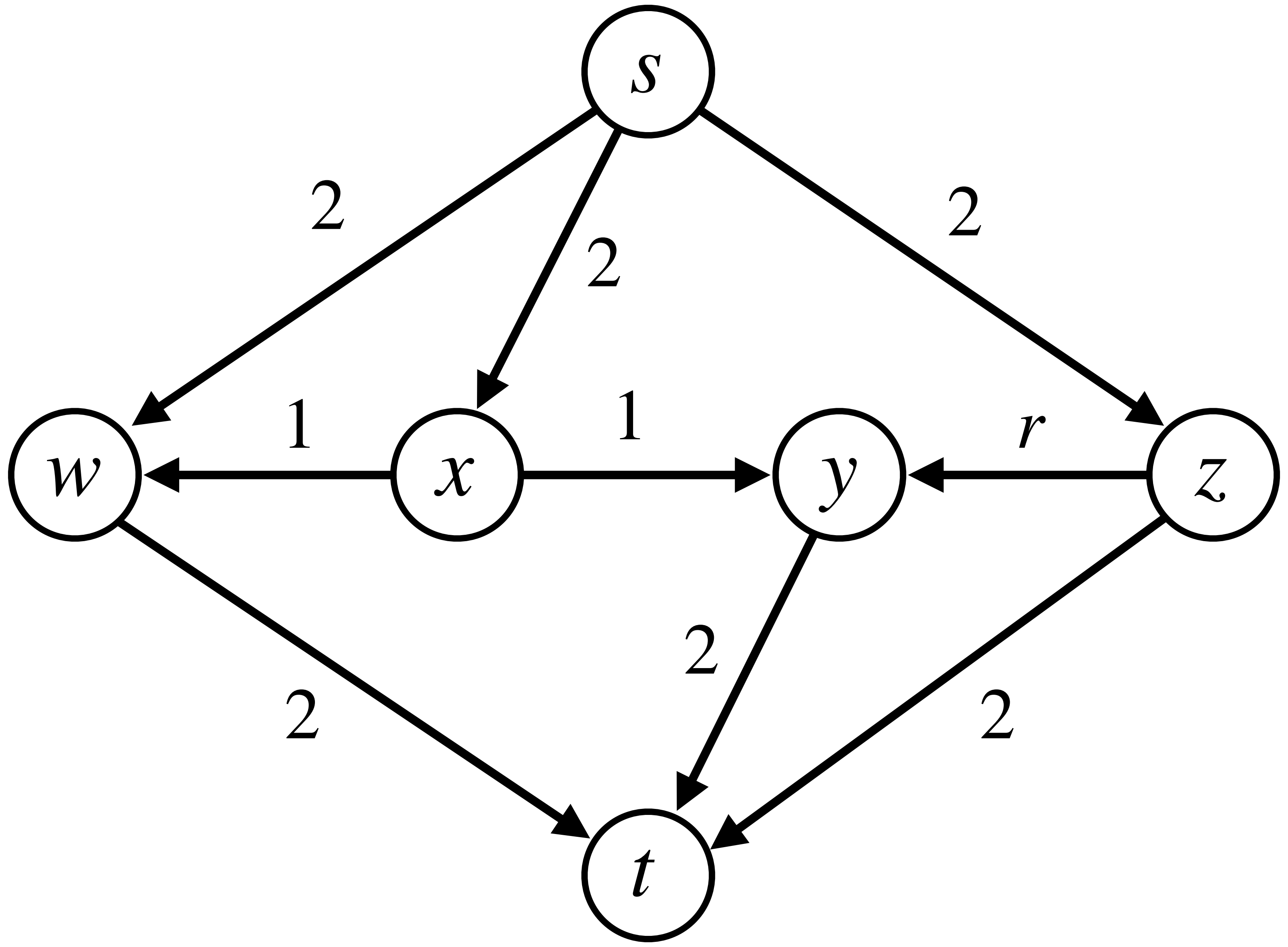
Ford-Fulkerson Method: A Non-terminating Case

Ford-Fulkerson may not terminate when some capacities are irrational.

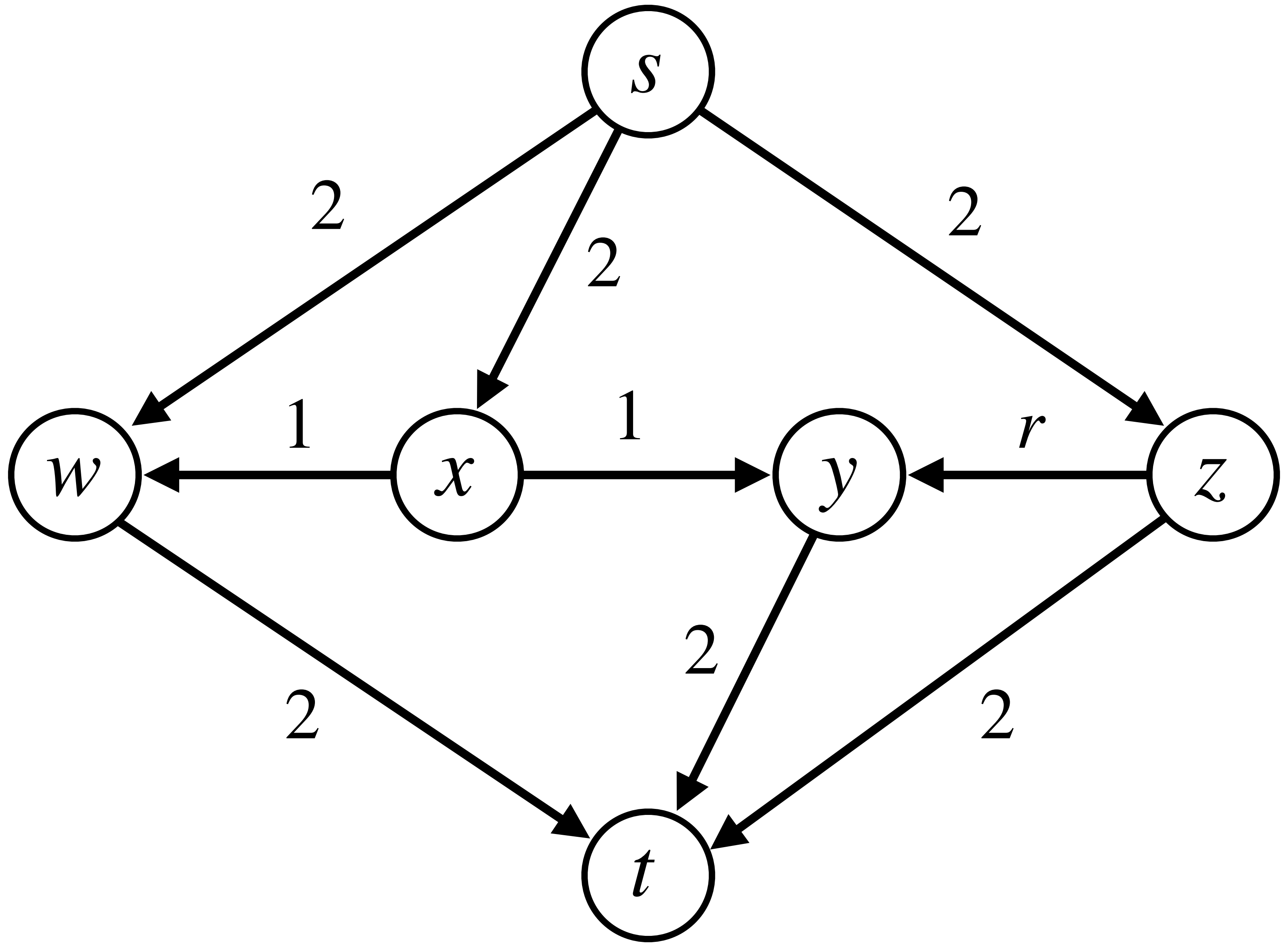
A famous example is on the next slide.

Ford-Fulkerson Method: A Non-terminating Case

Ford-Fulkerson Method: A Non-terminating Case



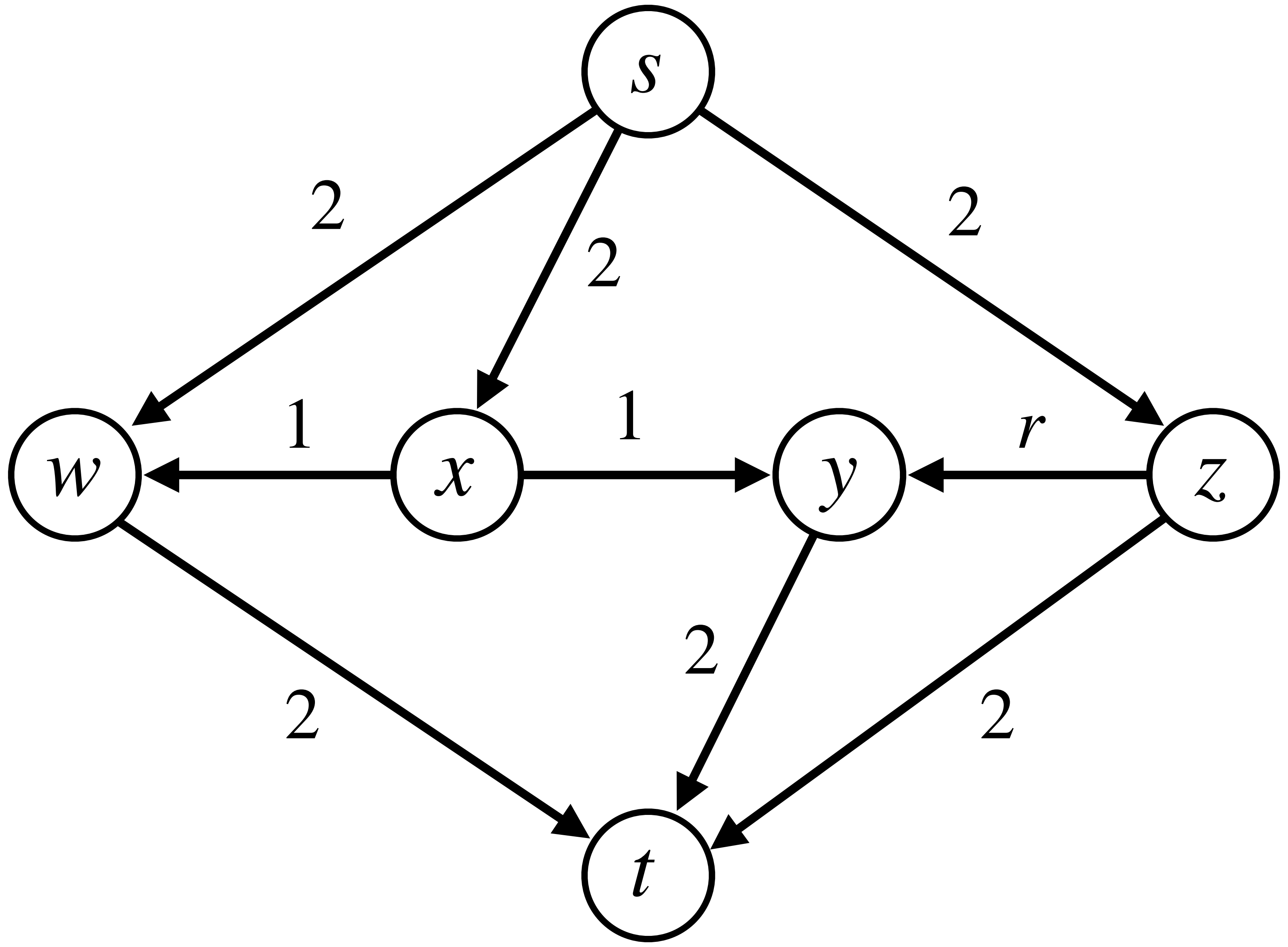
Ford-Fulkerson Method: A Non-terminating Case



$r = (\sqrt{5} - 1)/2$ is chosen so that $r^2 = 1 - r$

Ford-Fulkerson Method: A Non-terminating Case

Let $P = \langle s, x, y, z \rangle$, $P_1 = \langle s, z, y, x, w \rangle$, $P_2 = \langle s, w, y, z, t \rangle$, $P_3 = \langle w, x, y, t \rangle$ in residual networks.



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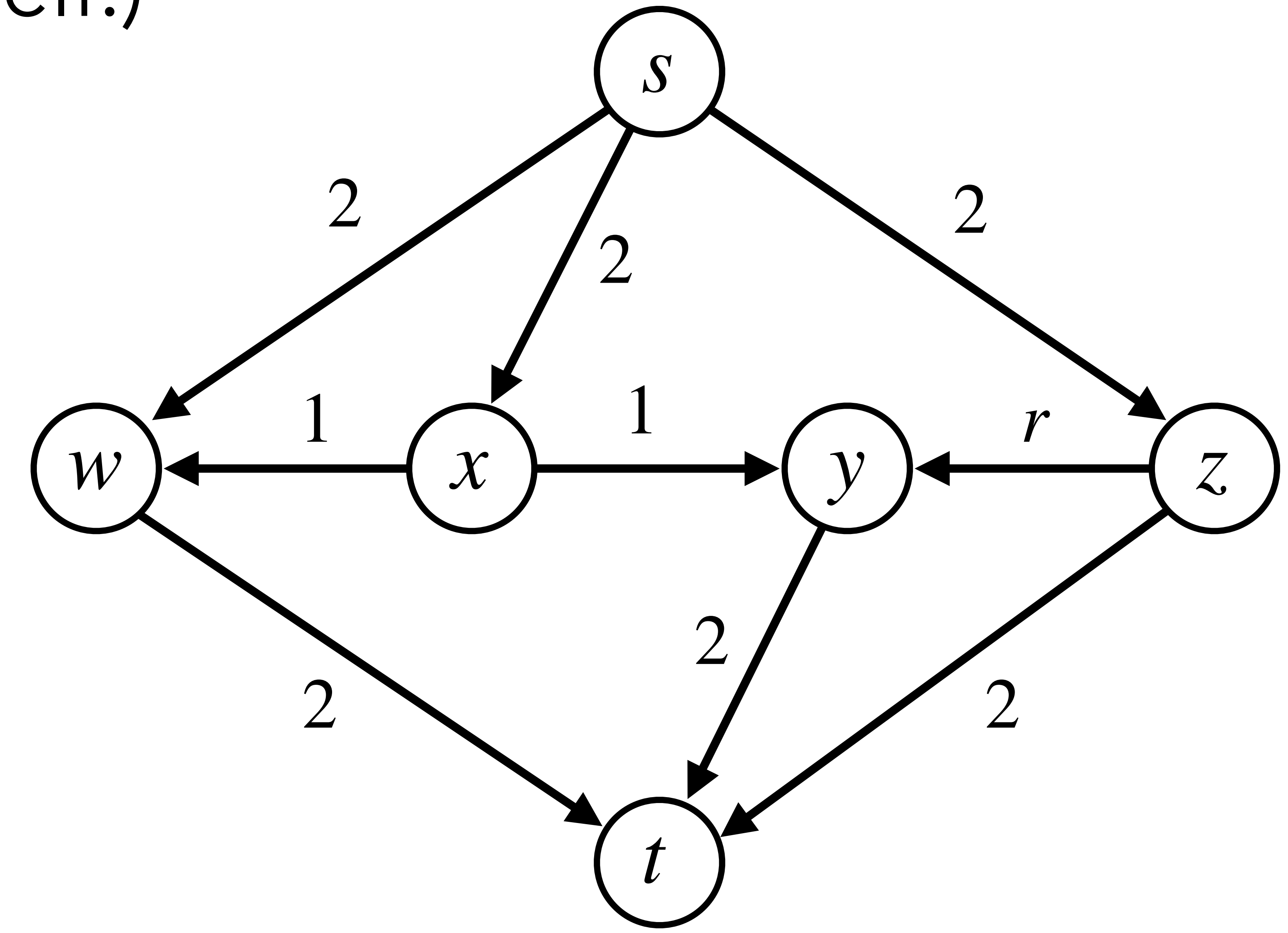
Then we can perform the following 5 steps: (Verify it yourself.)



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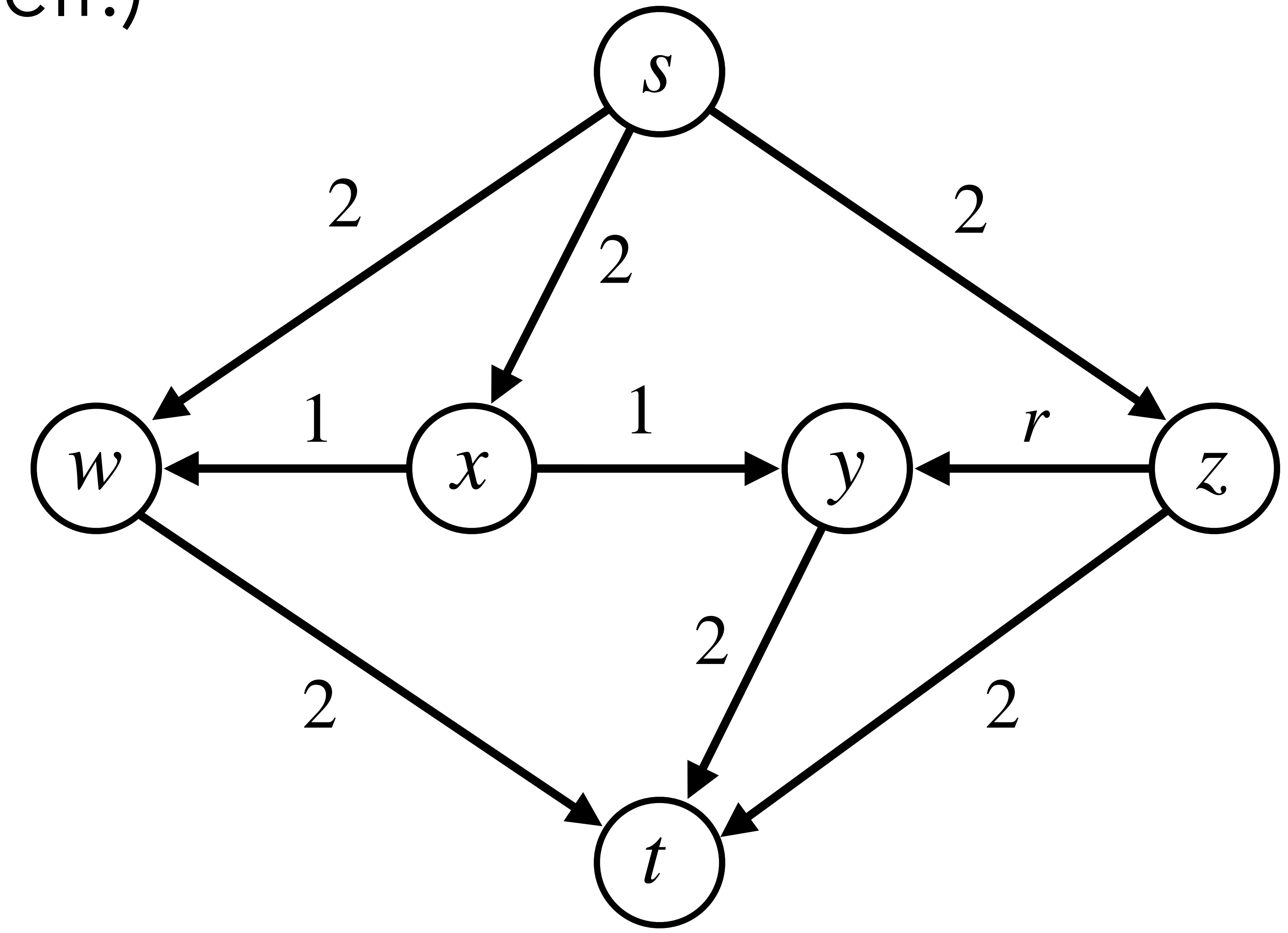
Step	Augmenting Path	Sent Flow
1		
2		
3		
4		
5		



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2		
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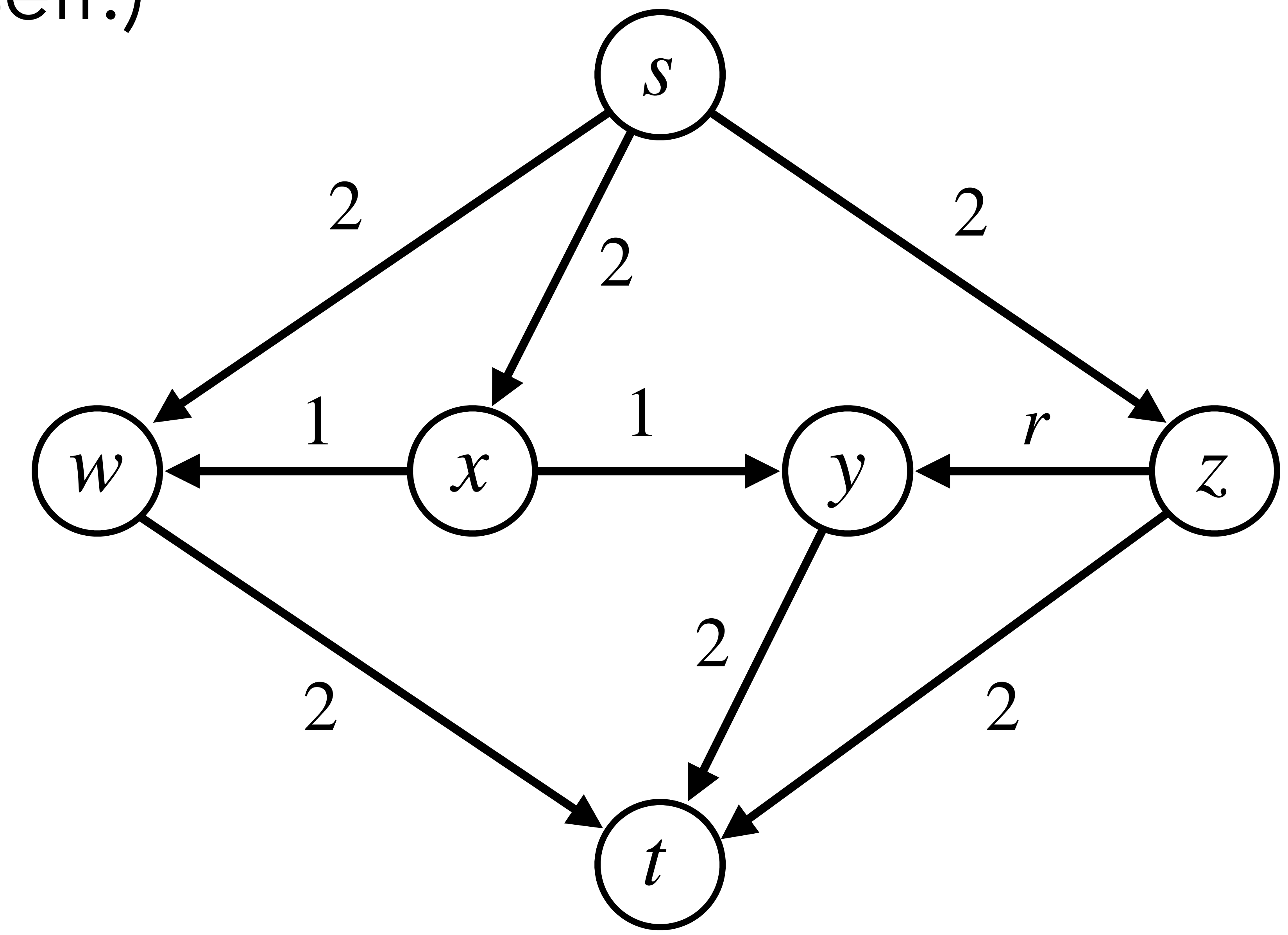

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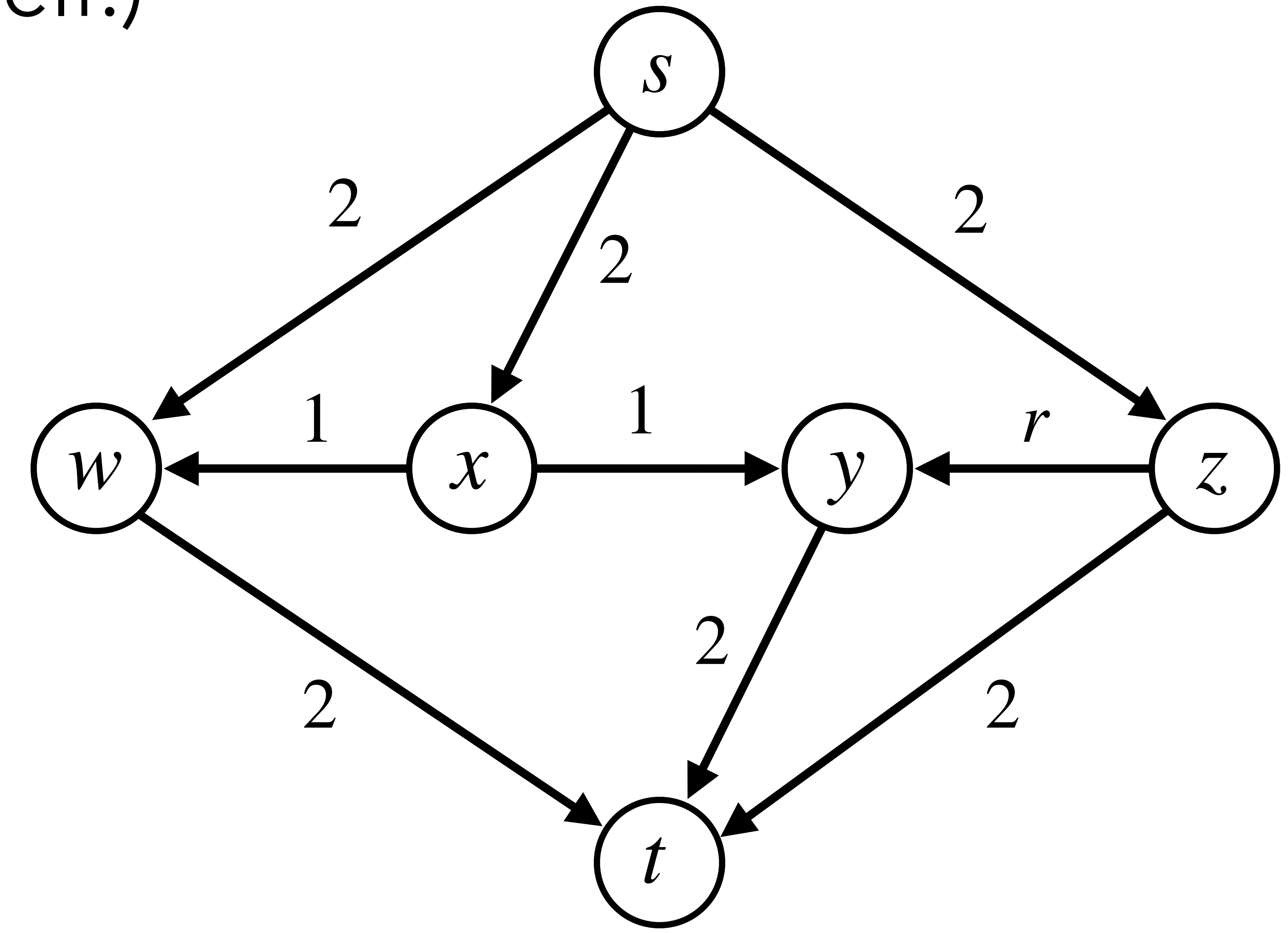
1	P	1
2	P_1	r
3		
4		
5		



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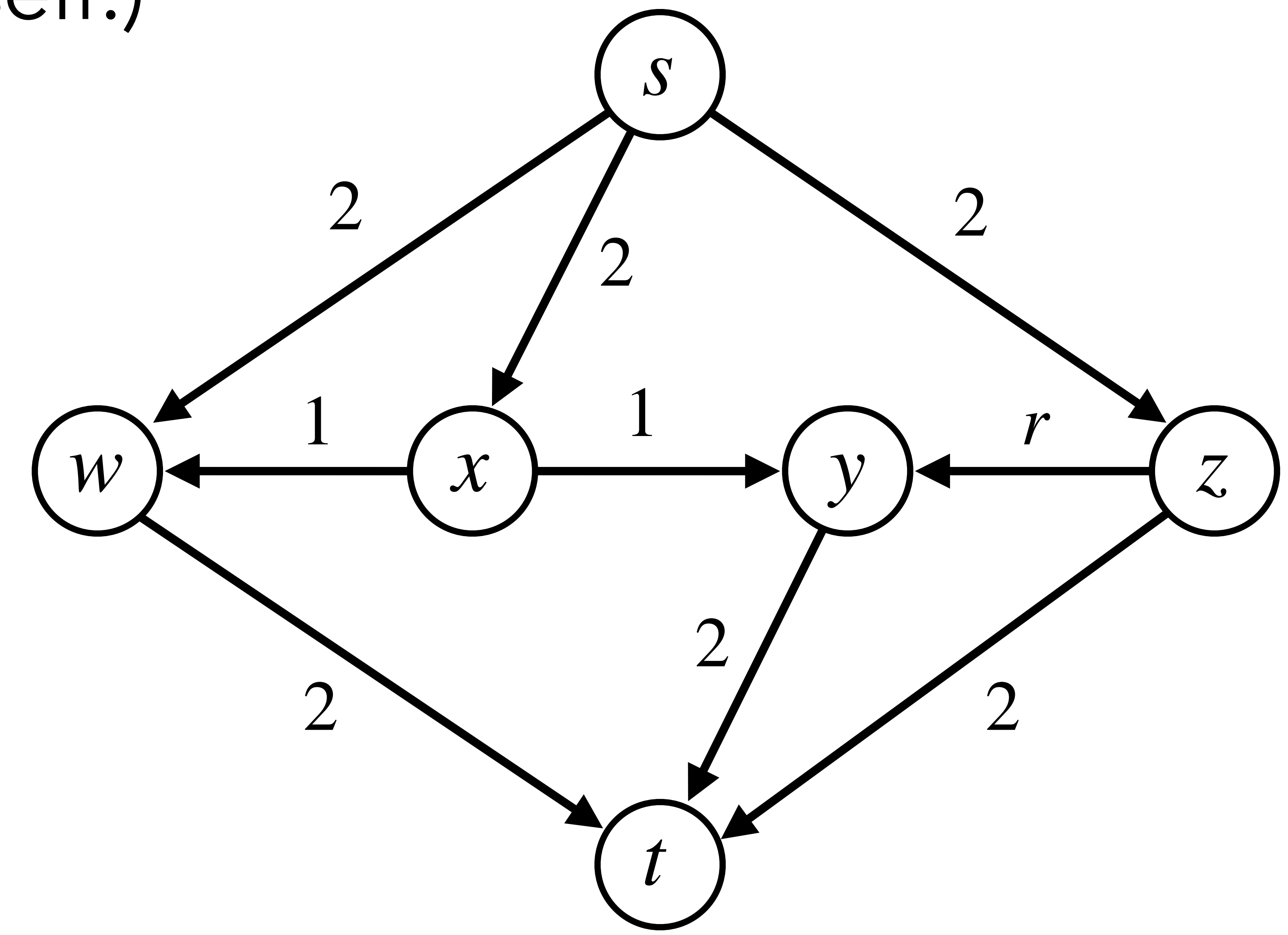
4		
5		



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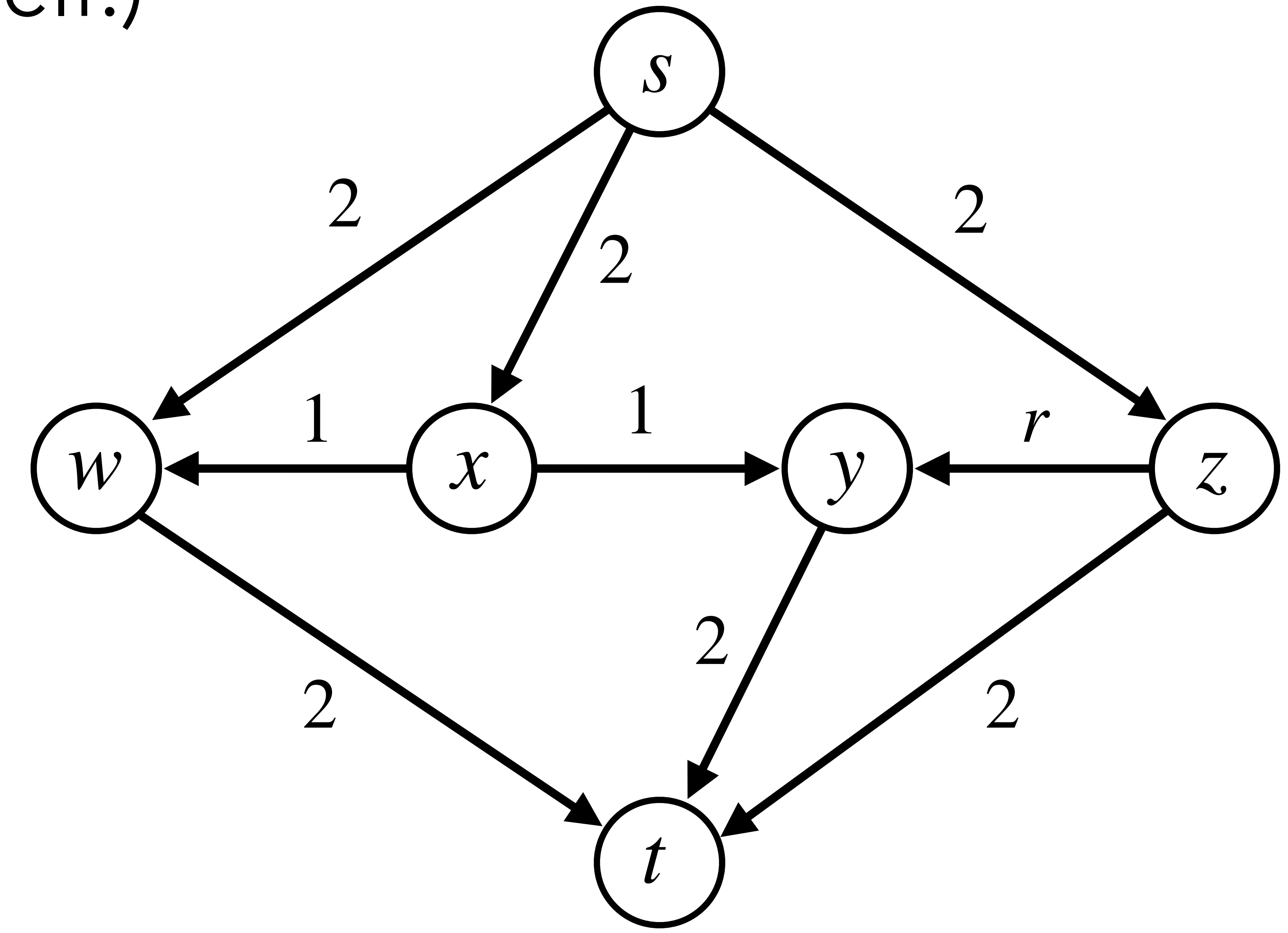
3	P_2	r
4	P_1	r^2
5		



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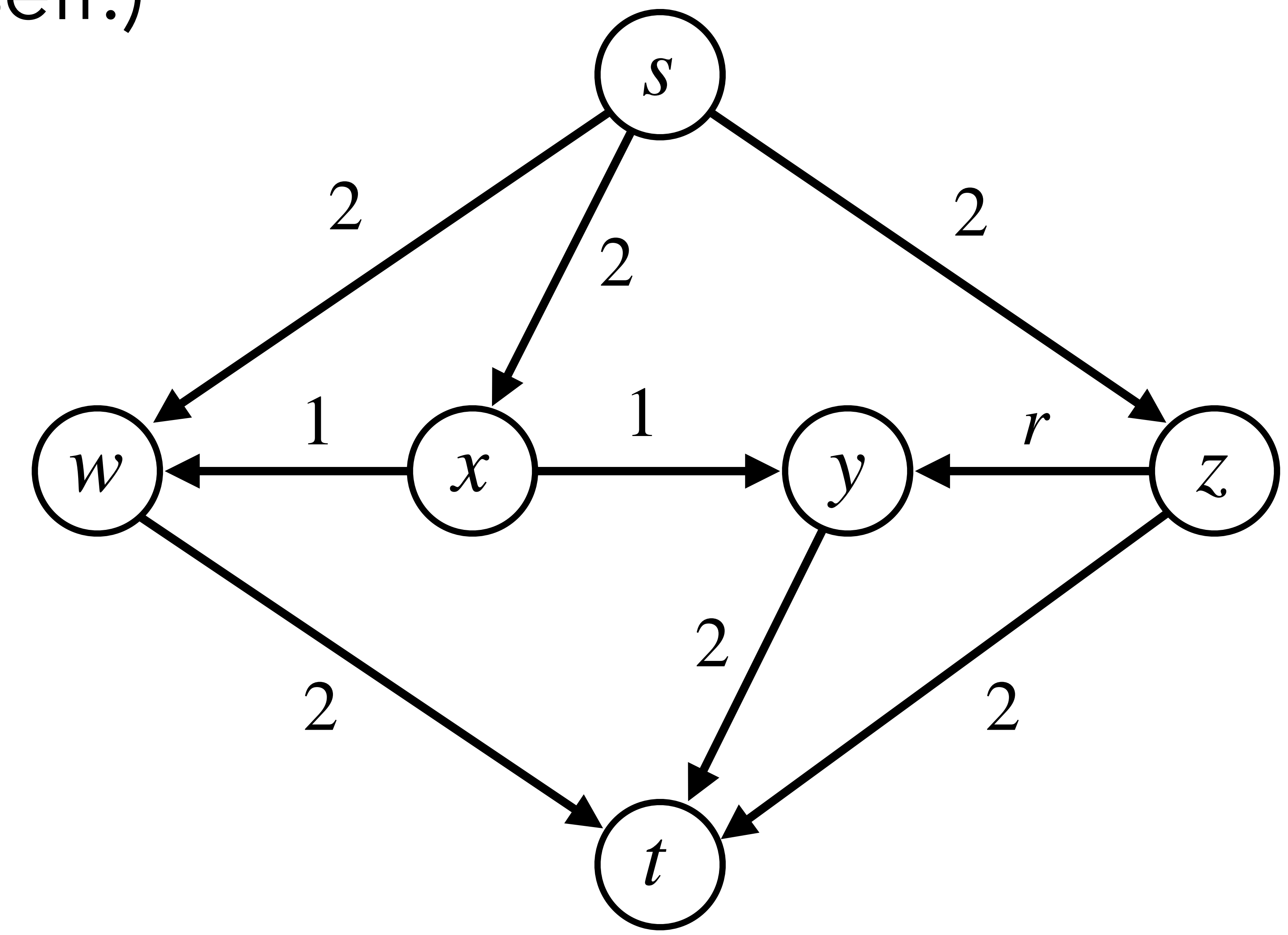
4	P_1	r^2
5	P_3	r^2



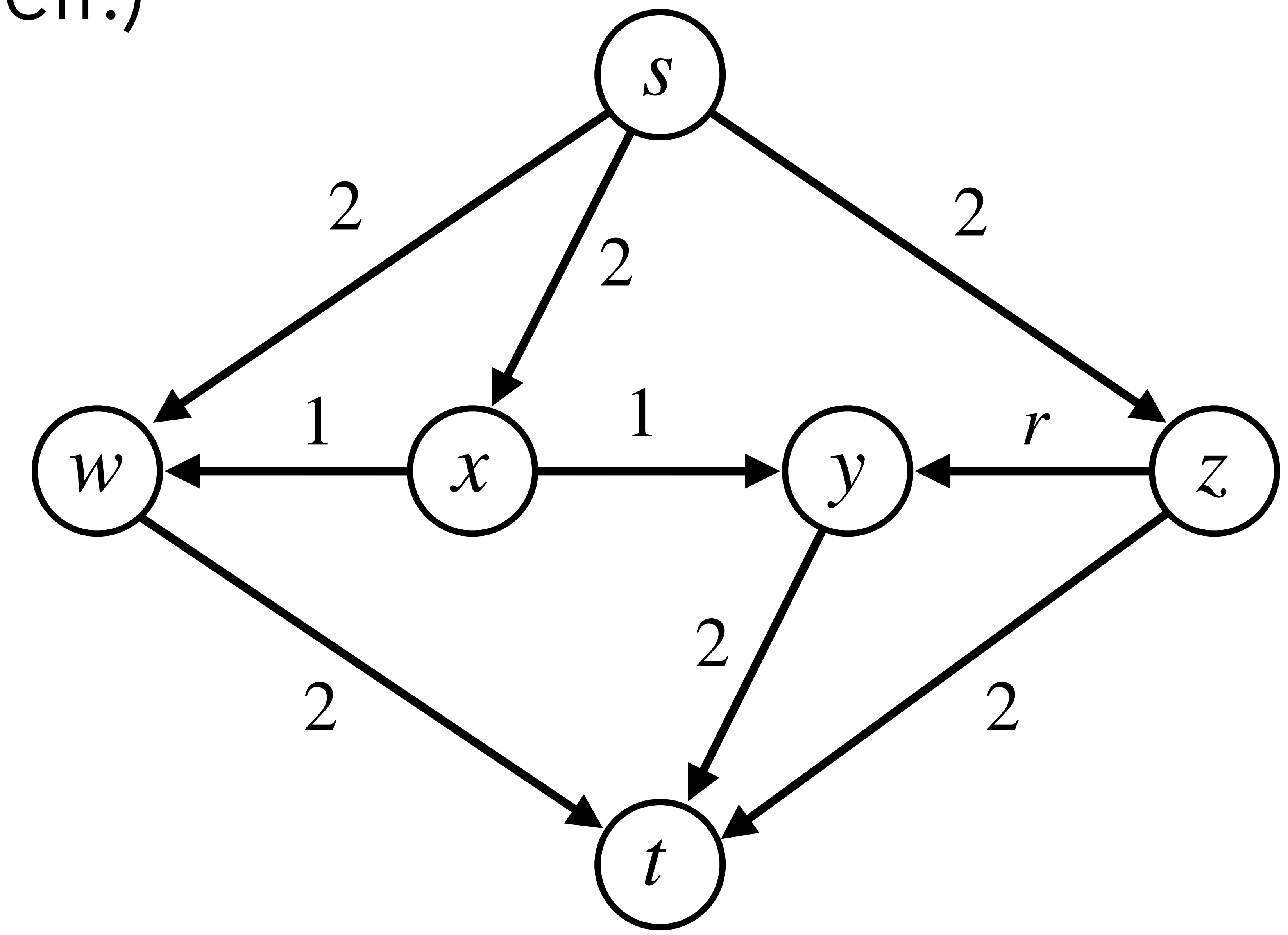
$r = (\sqrt{5} - 1)/2$ is chosen so that $r^2 = 1 - r$

Ford-Fulkerson Method: A Non-terminating Case

Let $P = \langle s, x, y, z \rangle$, $P_1 = \langle s, z, y, x, w \rangle$, $P_2 = \langle s, w, y, z, t \rangle$, $P_3 = \langle w, x, y, t \rangle$ in residual networks.

Then we can perform the following 5 steps: (Verify it yourself.)

Step	Augmenting Path	Sent Flow
1	P	1
2	P_1	r
3	P_2	r
4	P_1	r^2
5	P_3	r^2



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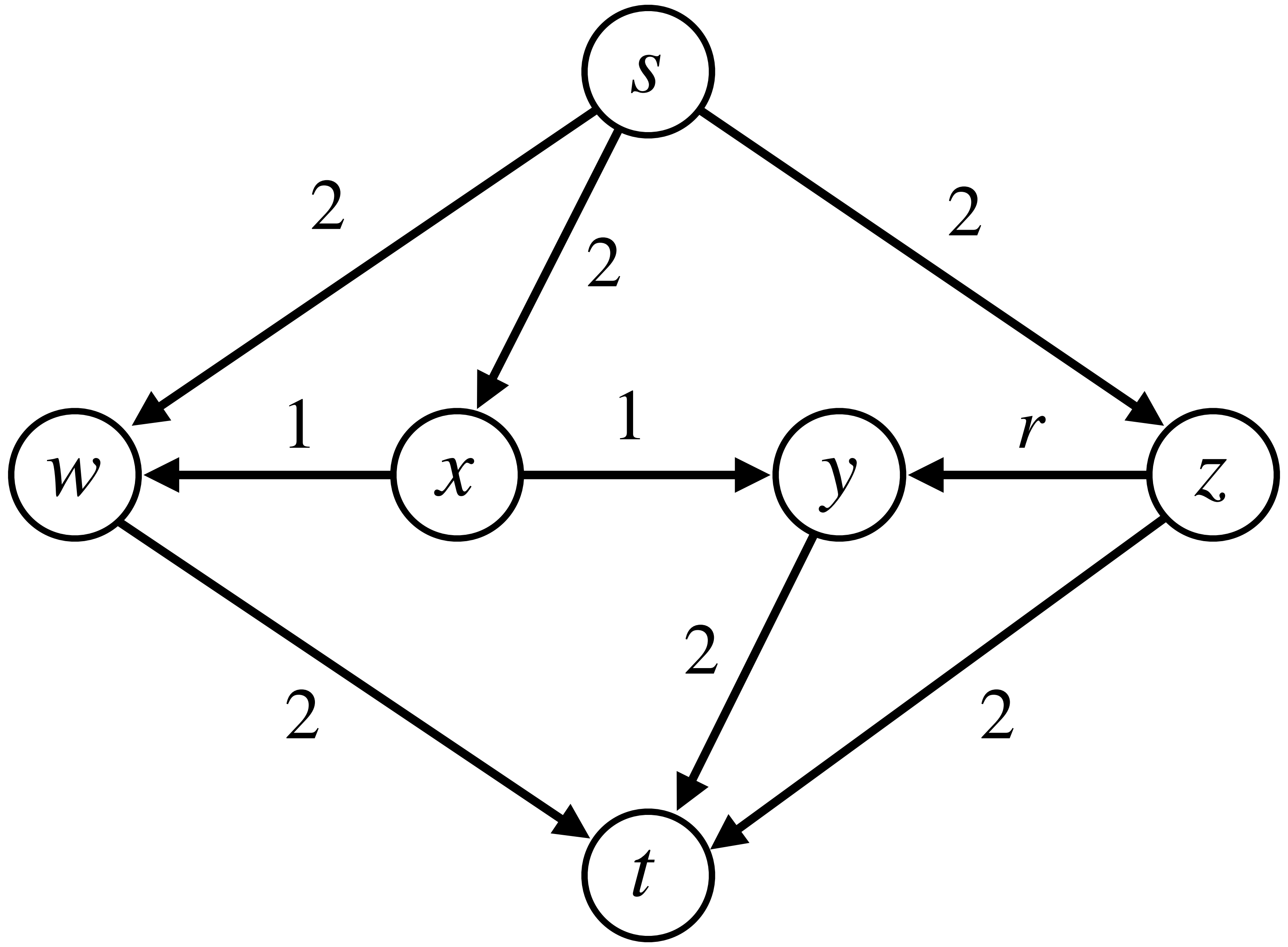
We can repeat **steps $2 - 5$** with flows r^3, r^3, r^4, and r^4 and keep doing so…

Ford-Fulkerson Method: A Non-terminating Case



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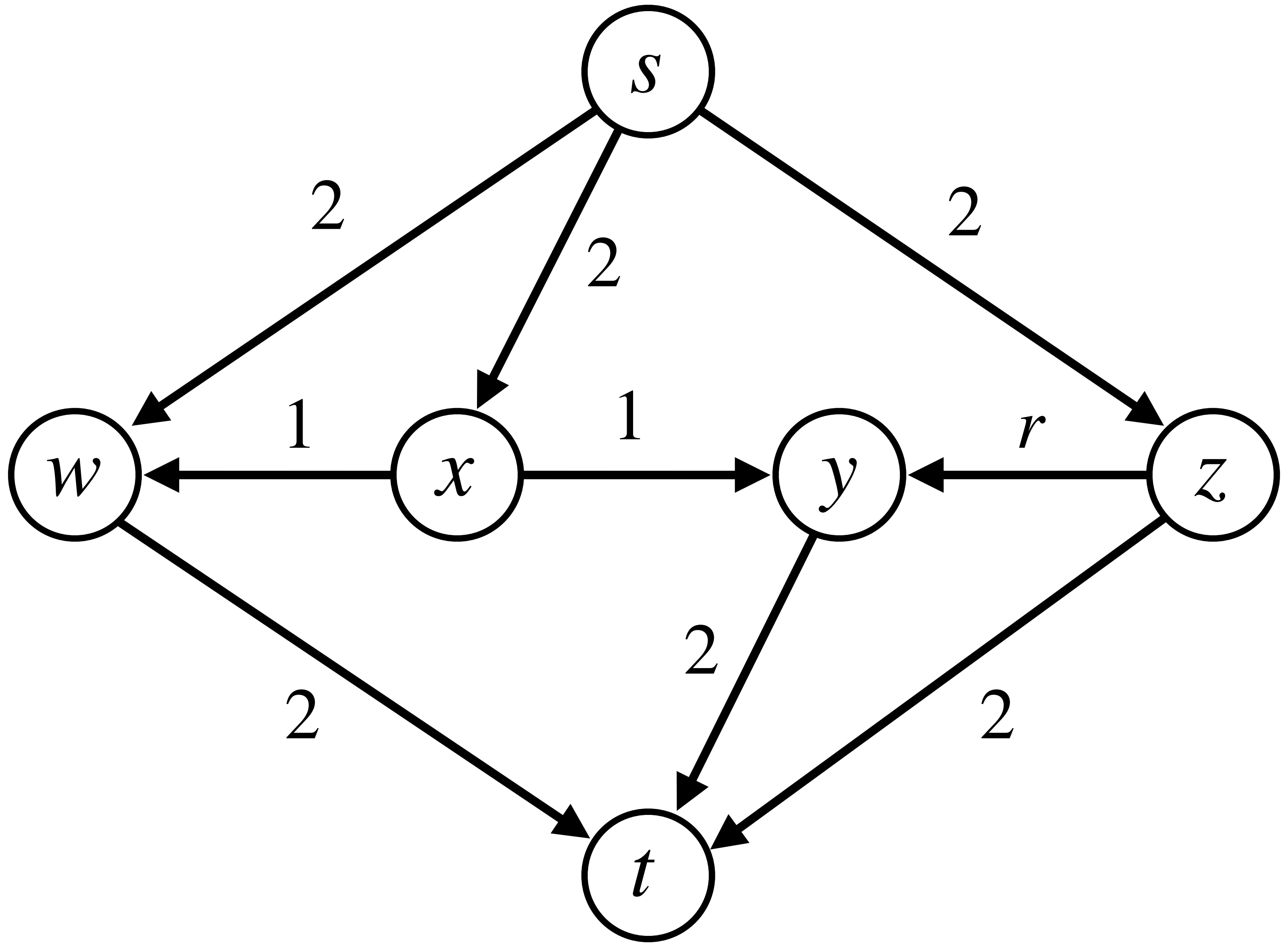
Ford-Fulkerson Method: A Non-terminating Case

The total flow will converge to



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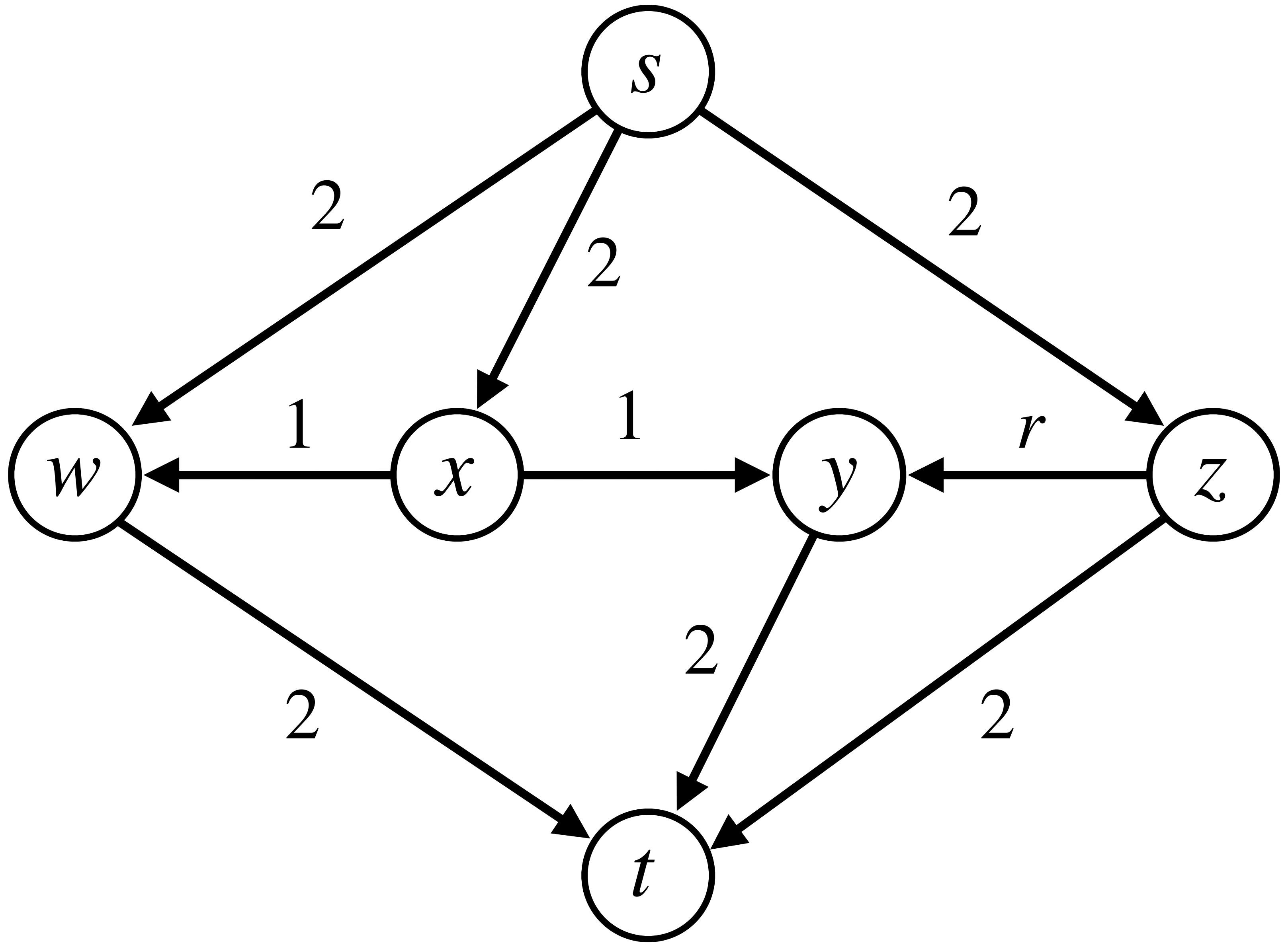
The total flow will converge to $1 + 2\sum_{i=1}^{\infty} r^i =$



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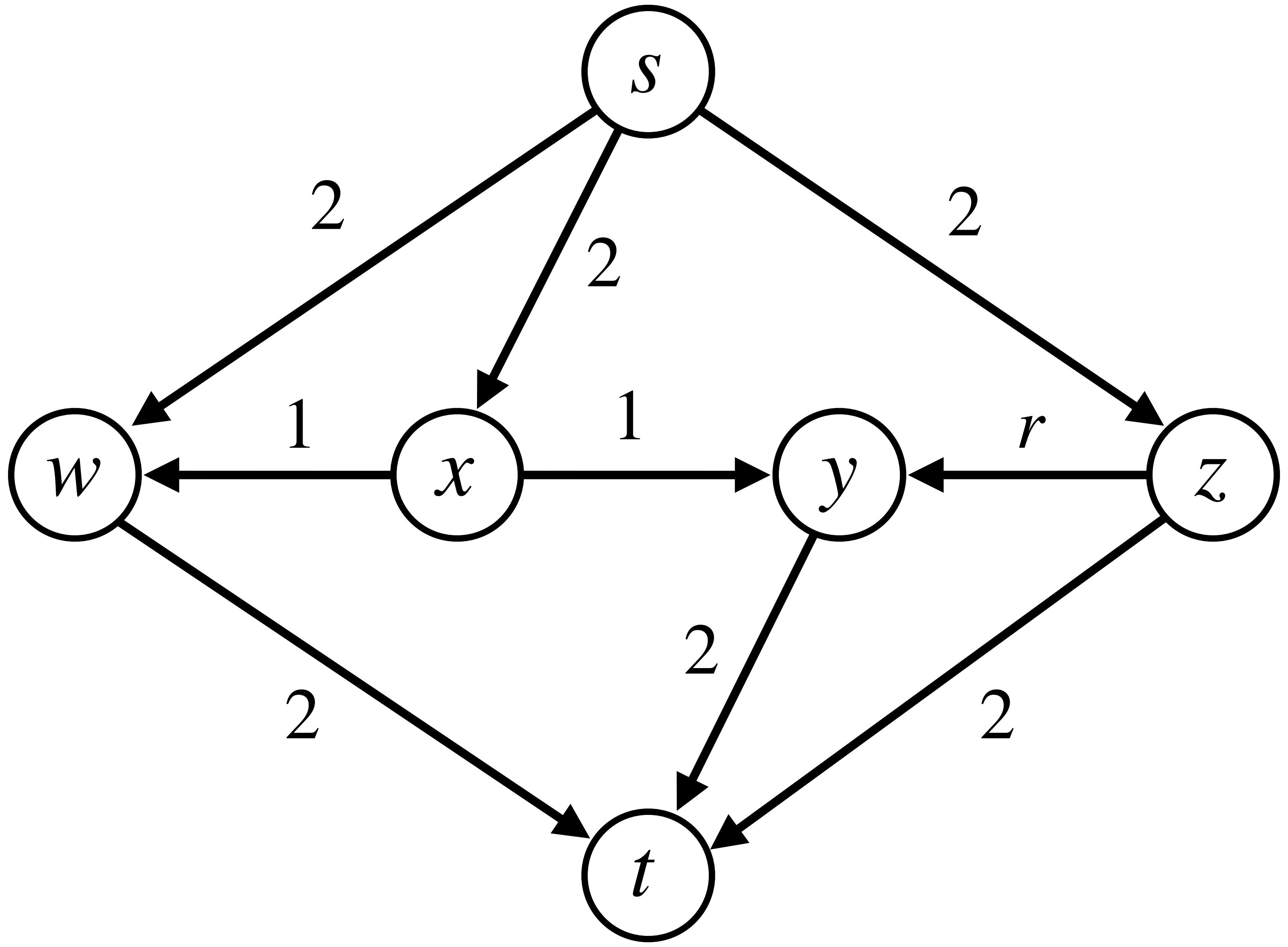
Ford-Fulkerson Method: A Non-terminating Case

The total flow will converge to $1 + 2\sum_{i=1}^{\infty} r^i = 2 + 3r$



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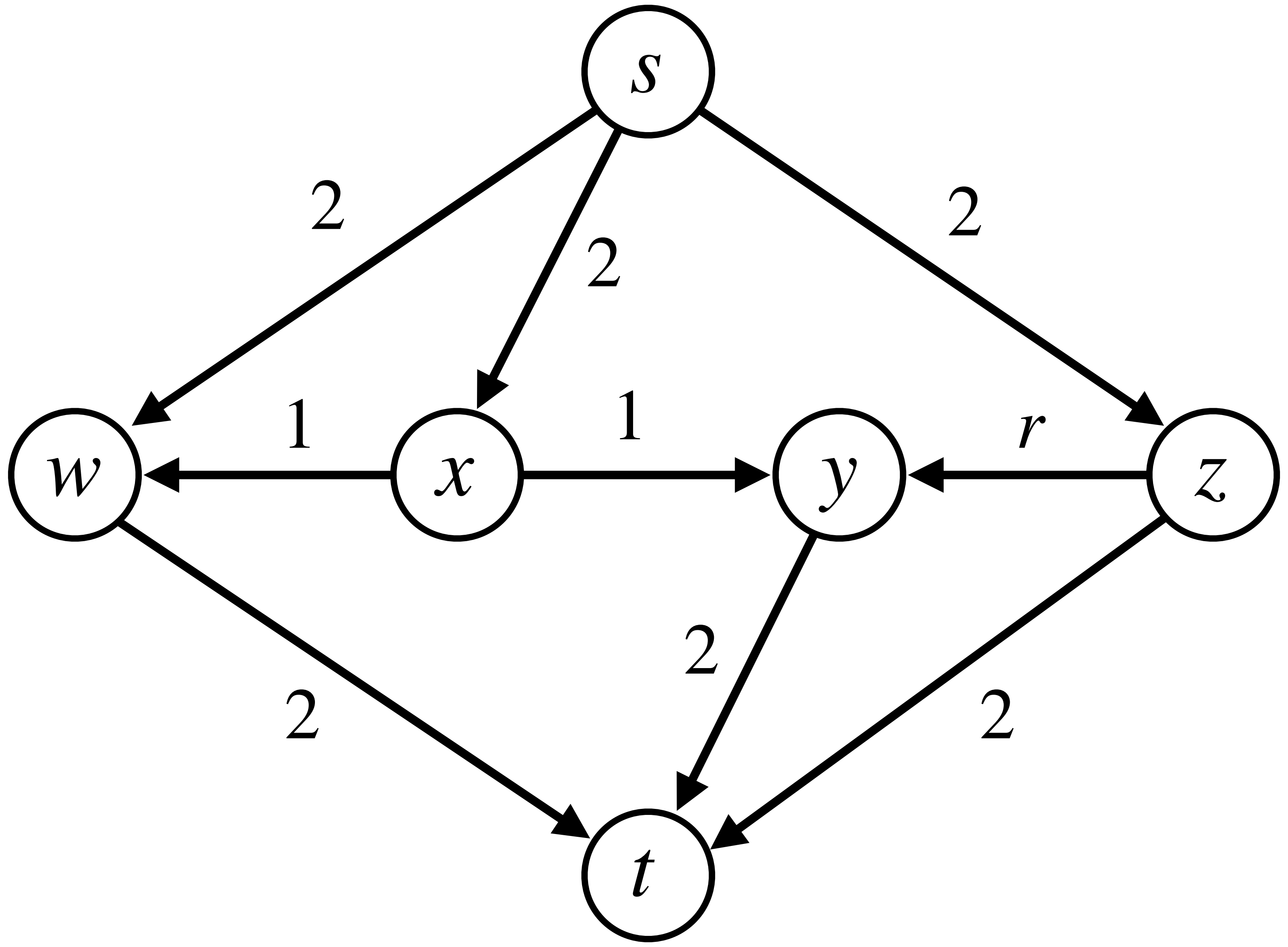
Ford-Fulkerson Method: A Non-terminating Case

The total flow will converge to $1 + 2\sum_{i=1}^{\infty} r^i = 2 + 3r < 5$.



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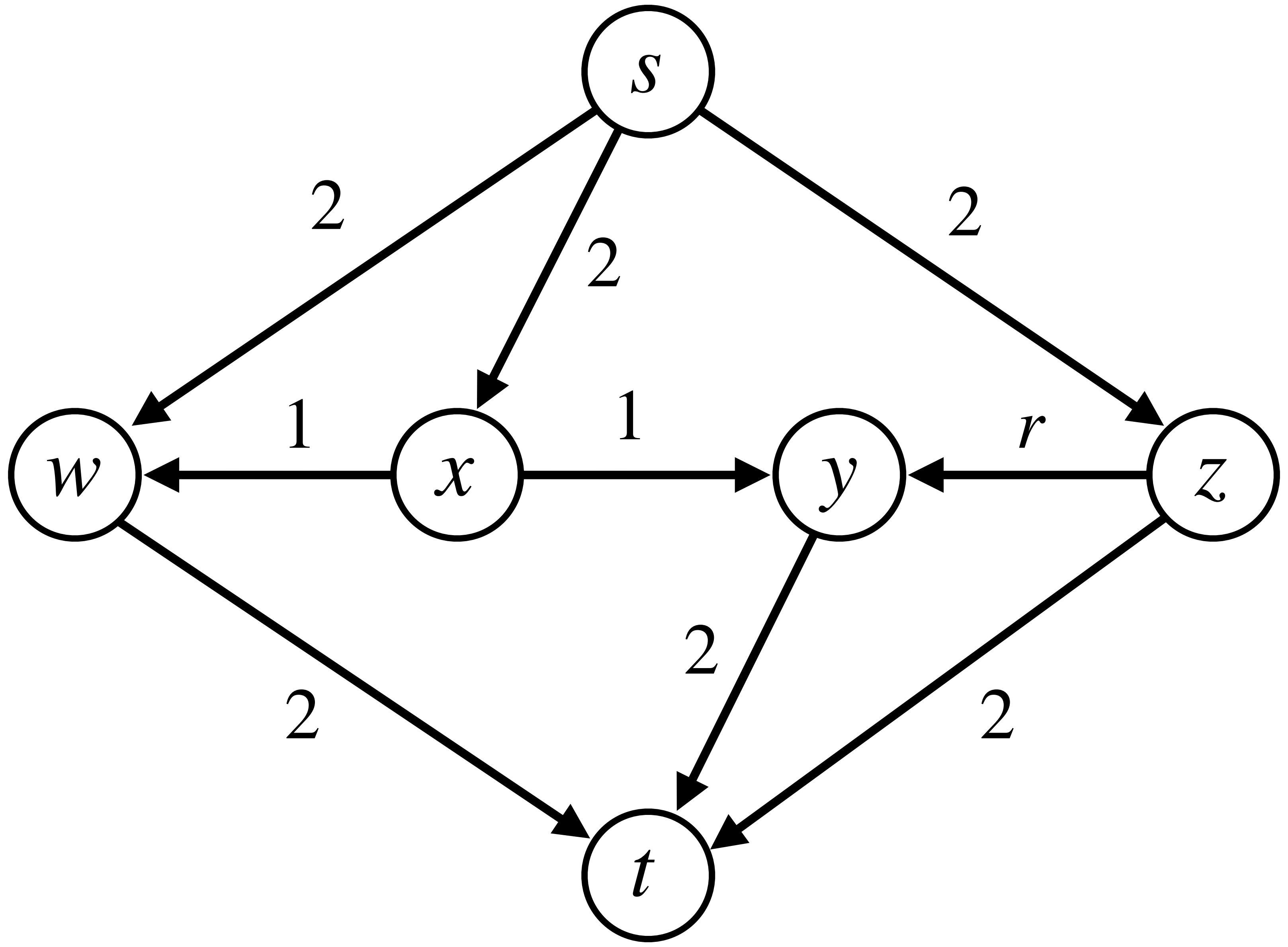
There is a flow with value 5 in the given network.



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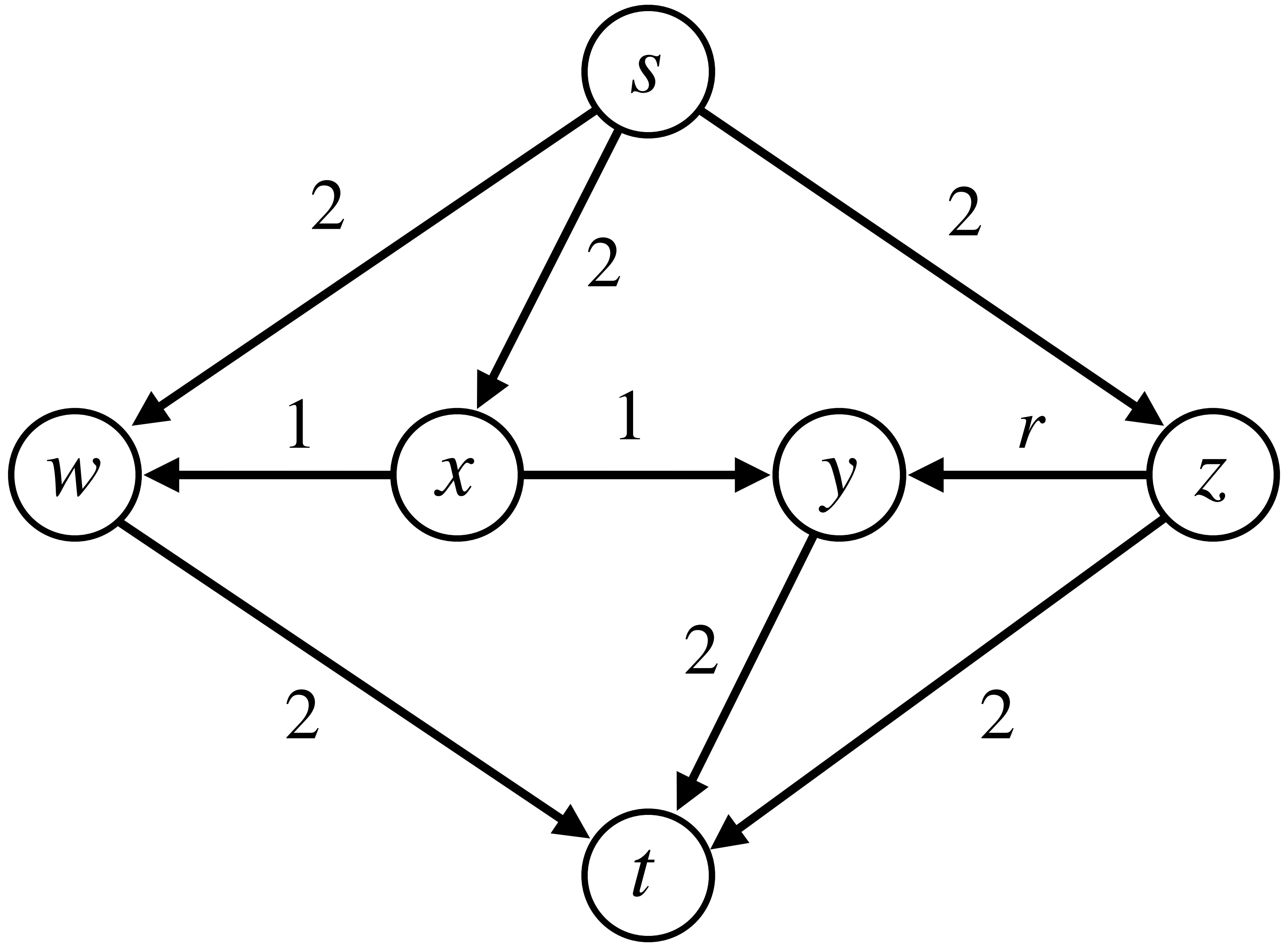
Hence, the algorithm will never terminate.



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Ford-Fulkerson Method

Will Ford-Fulkerson terminate when capacities are rationals?

Ford-Fulkerson Method

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Yes, prove it yourself.

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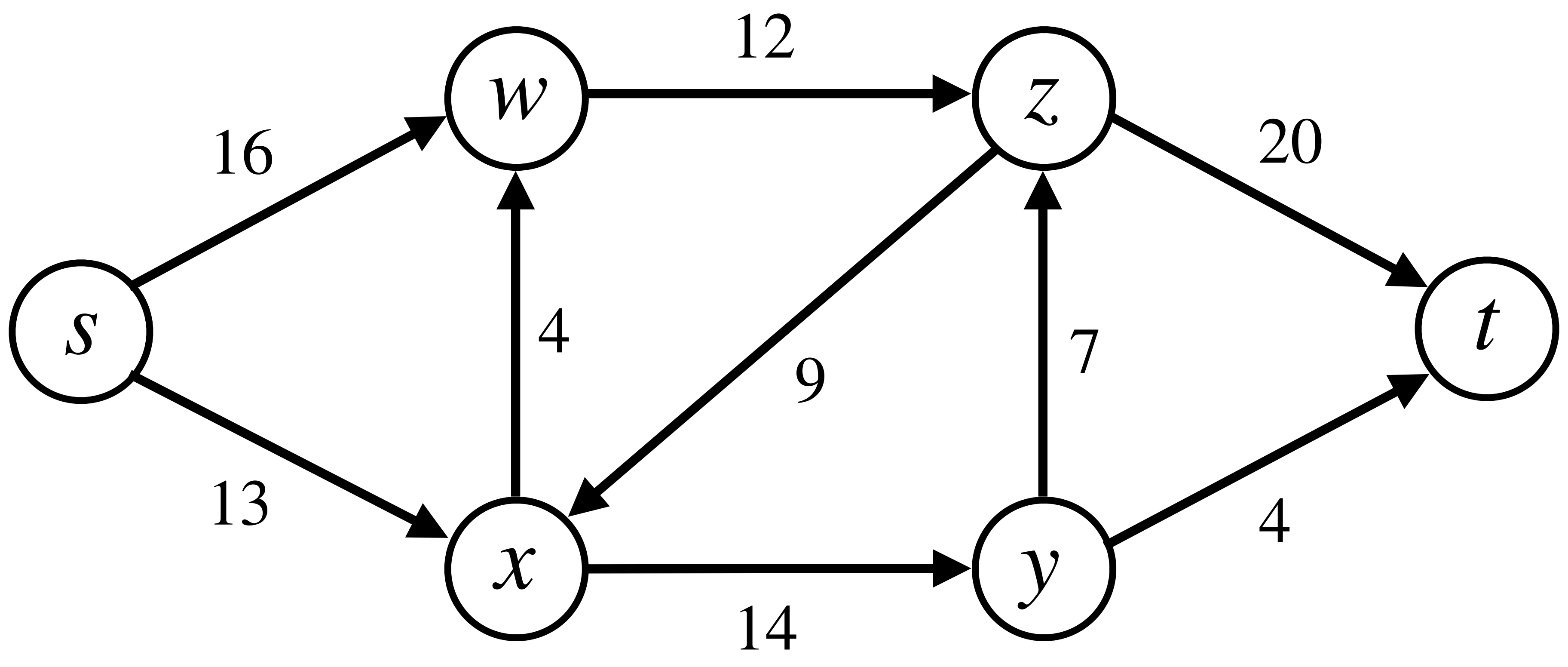

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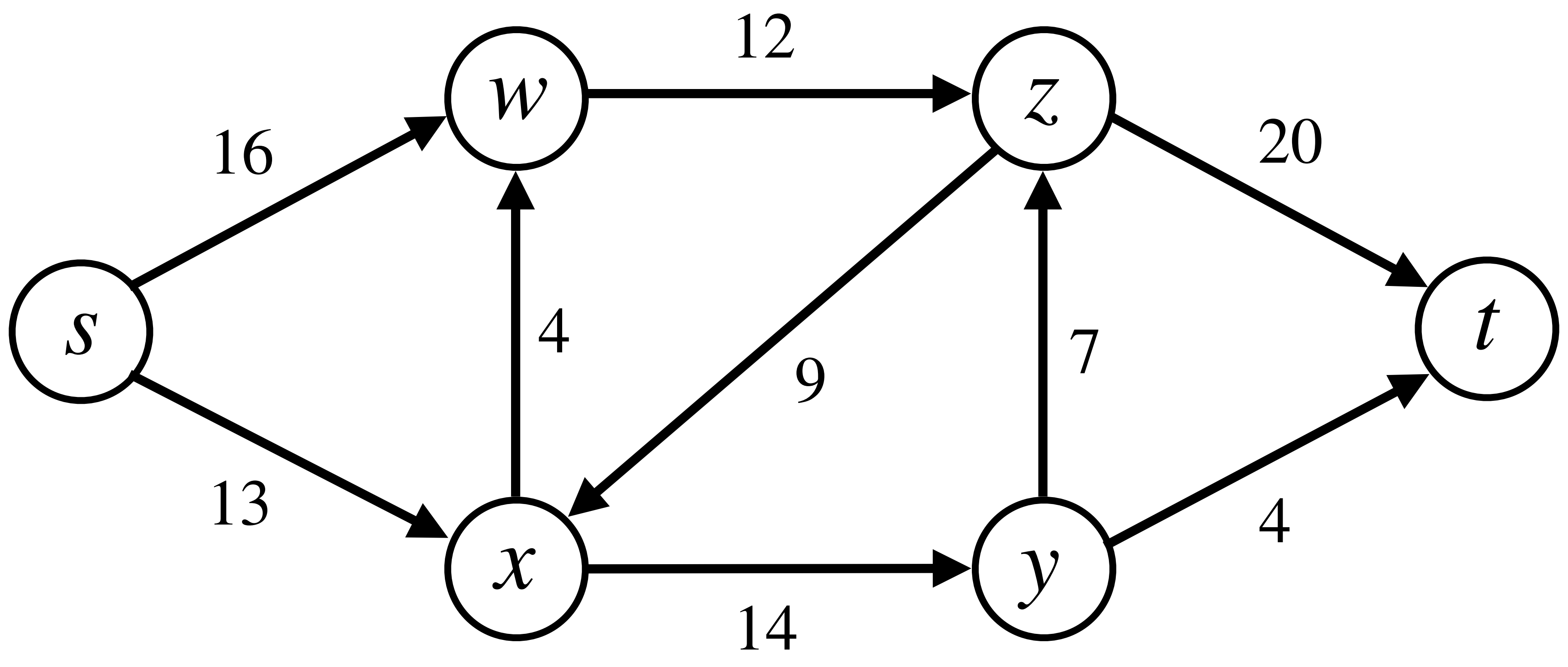

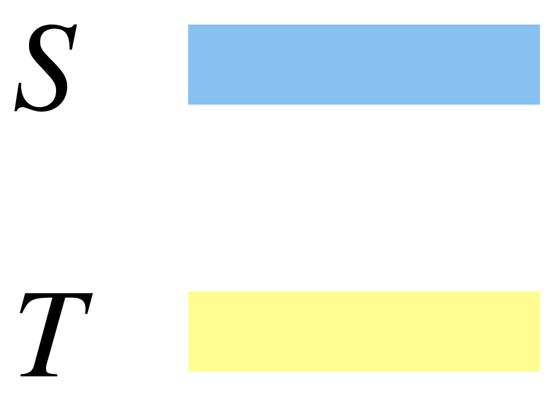

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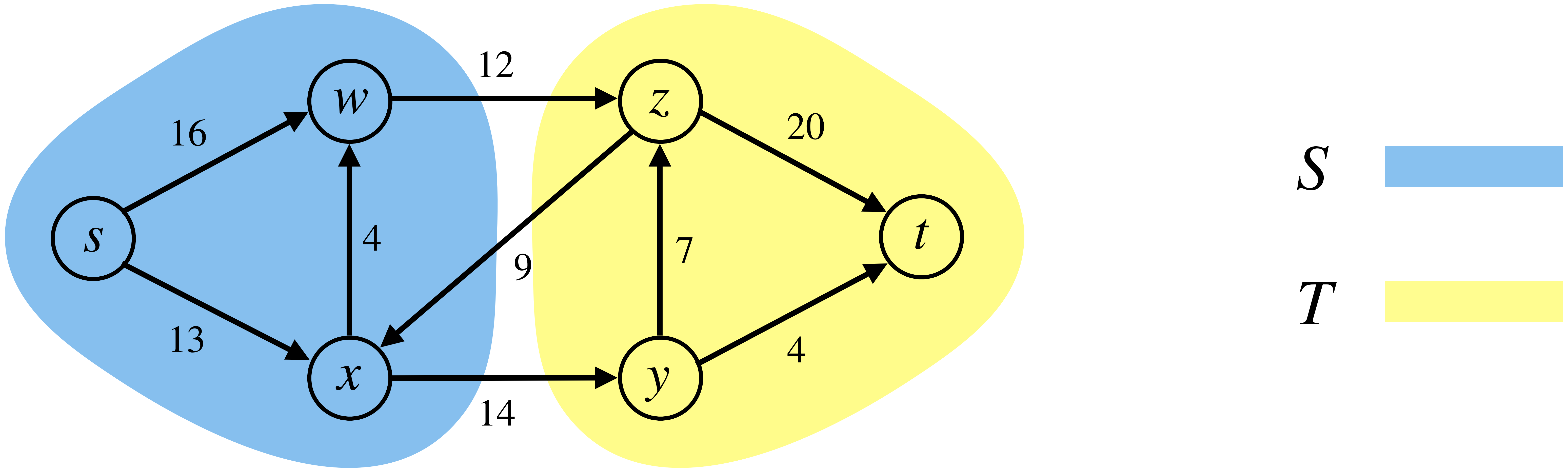

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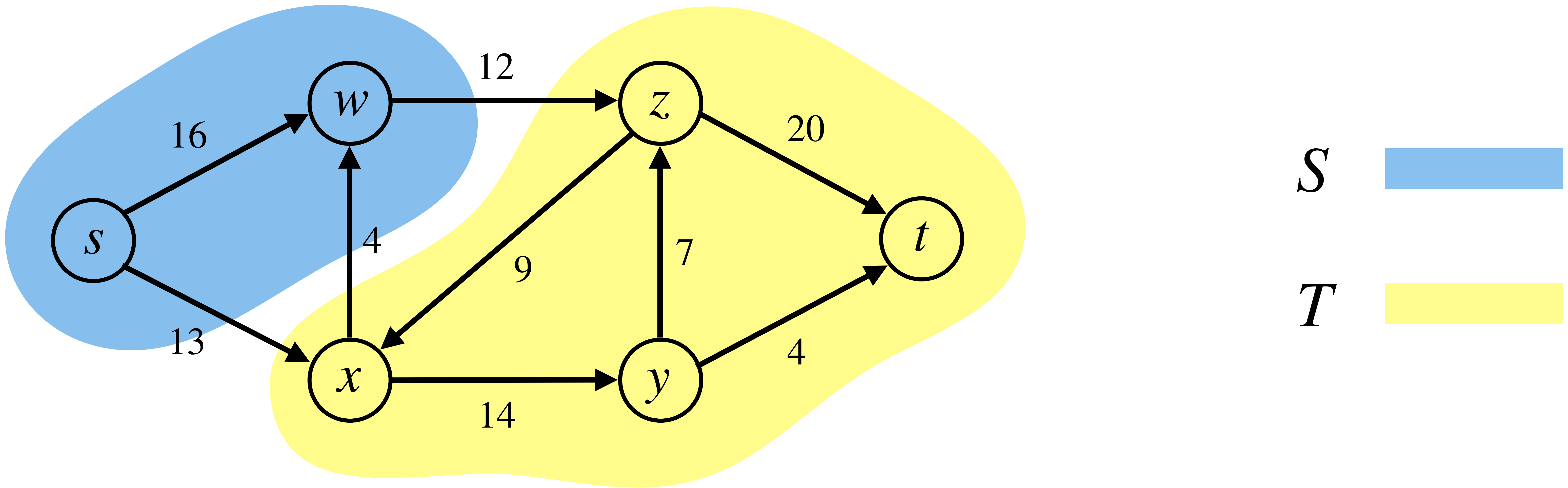

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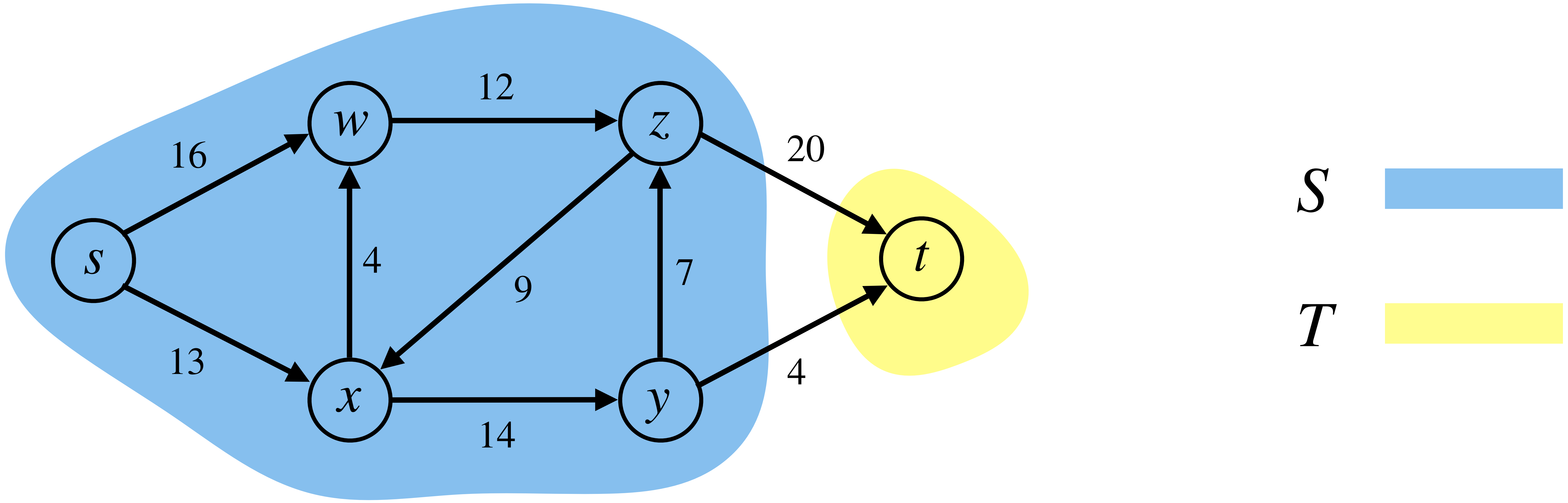

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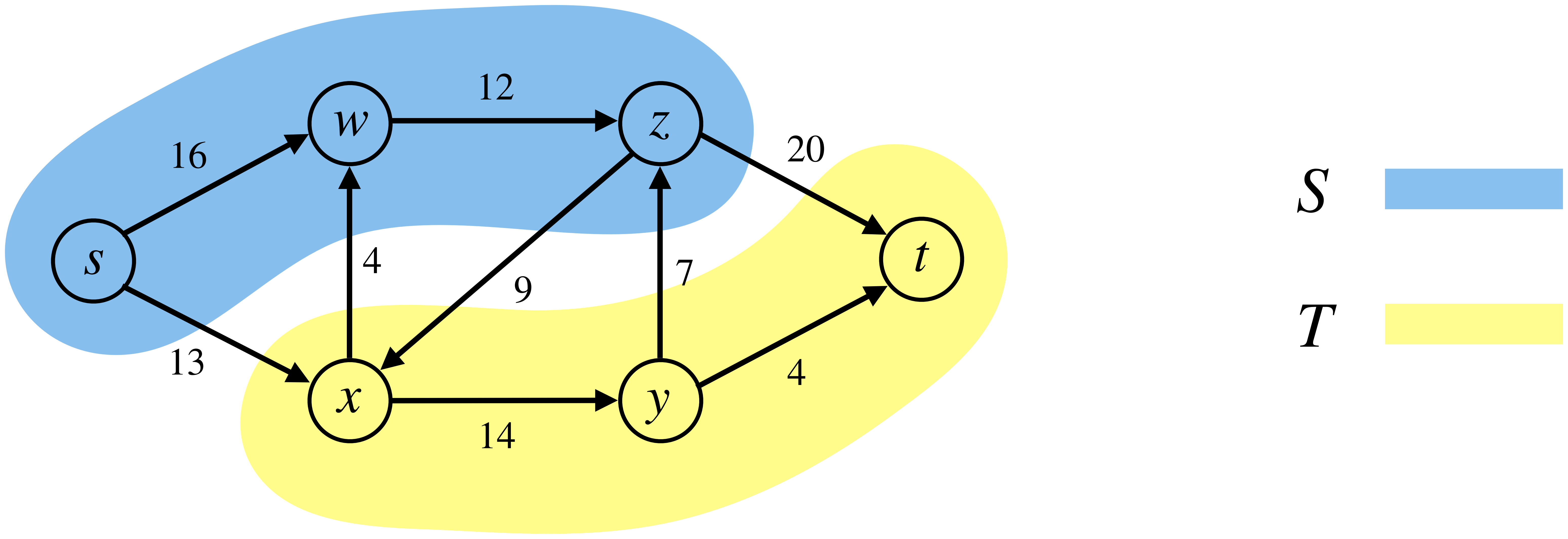

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Total flow from T to S

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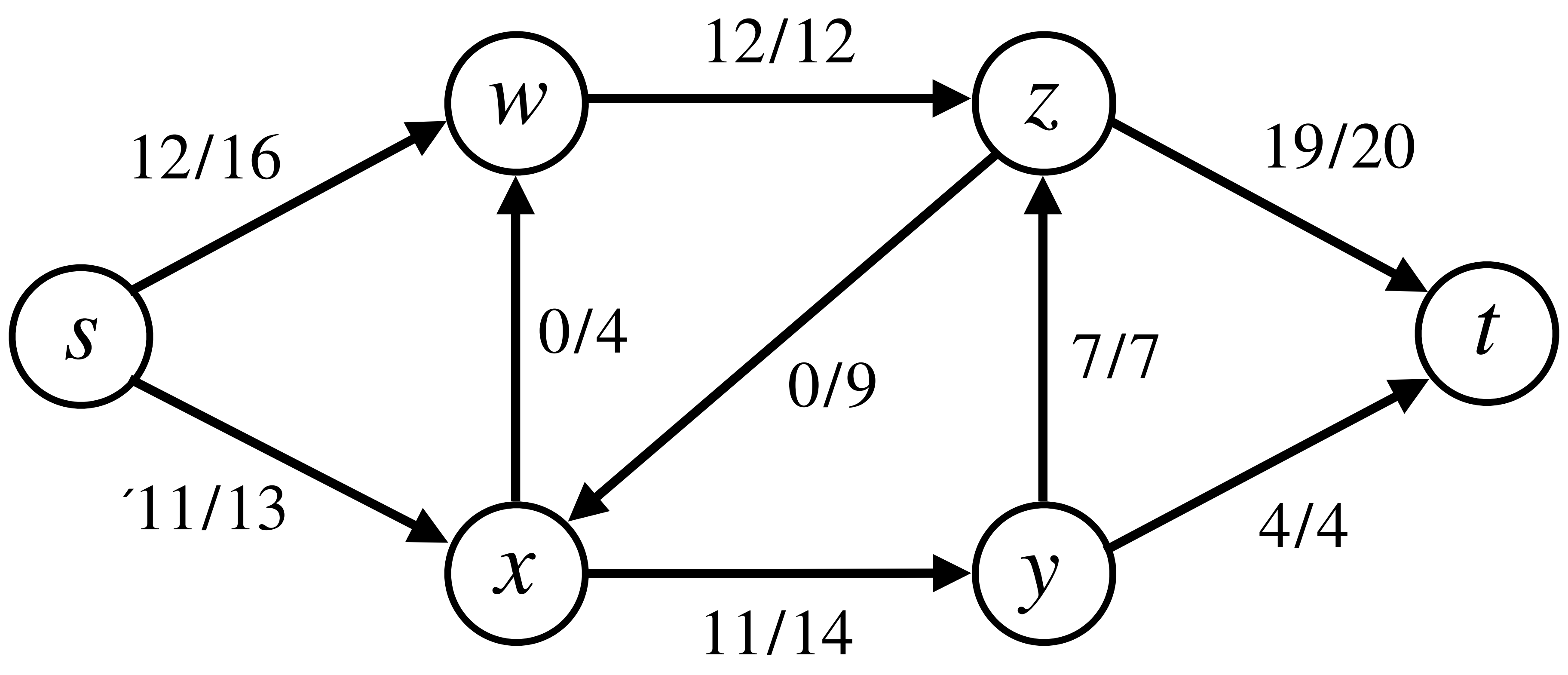

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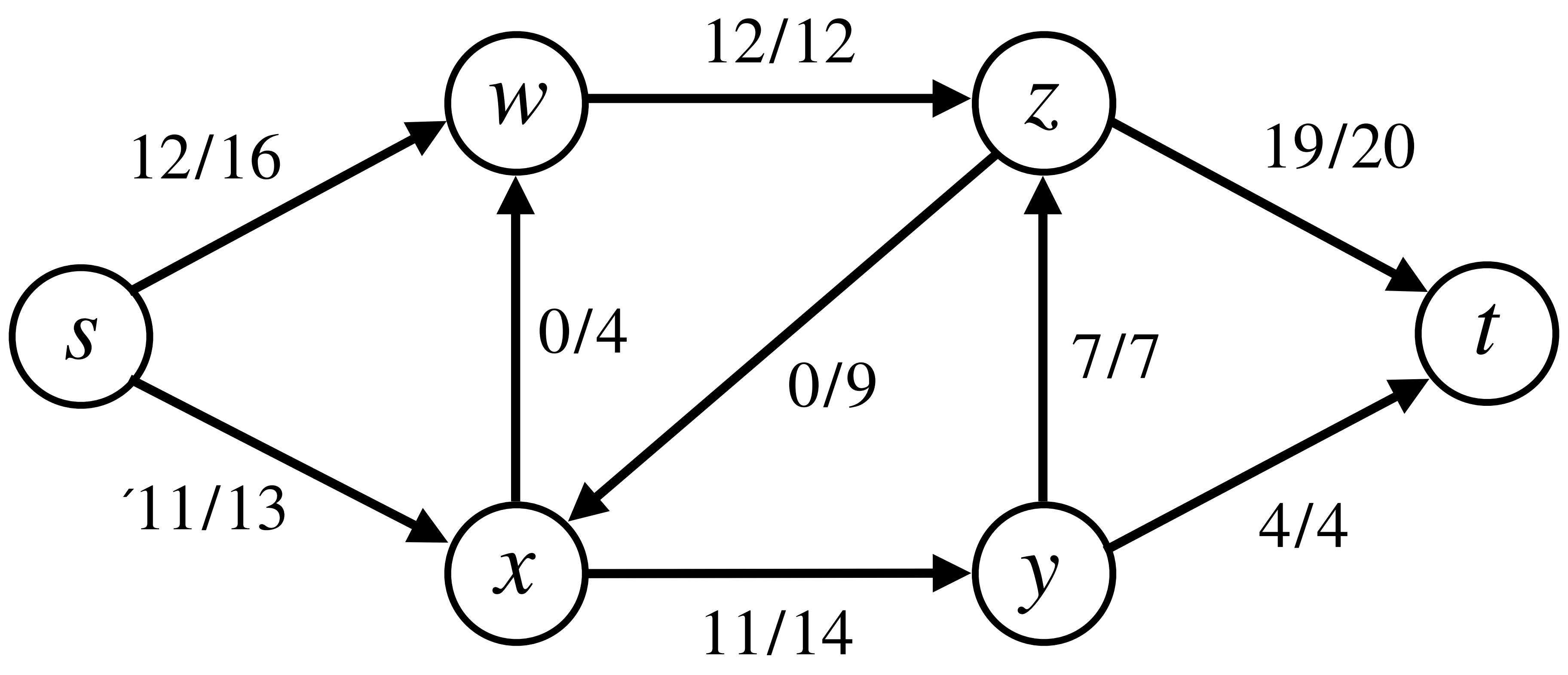

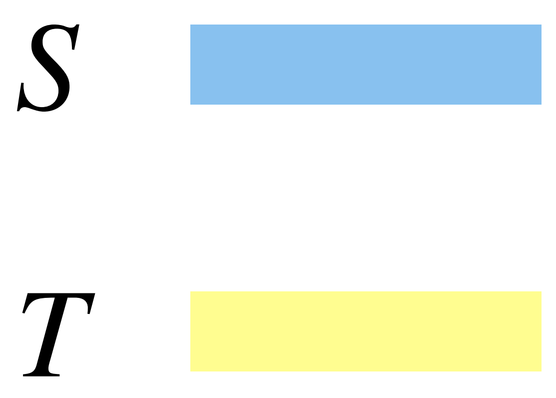

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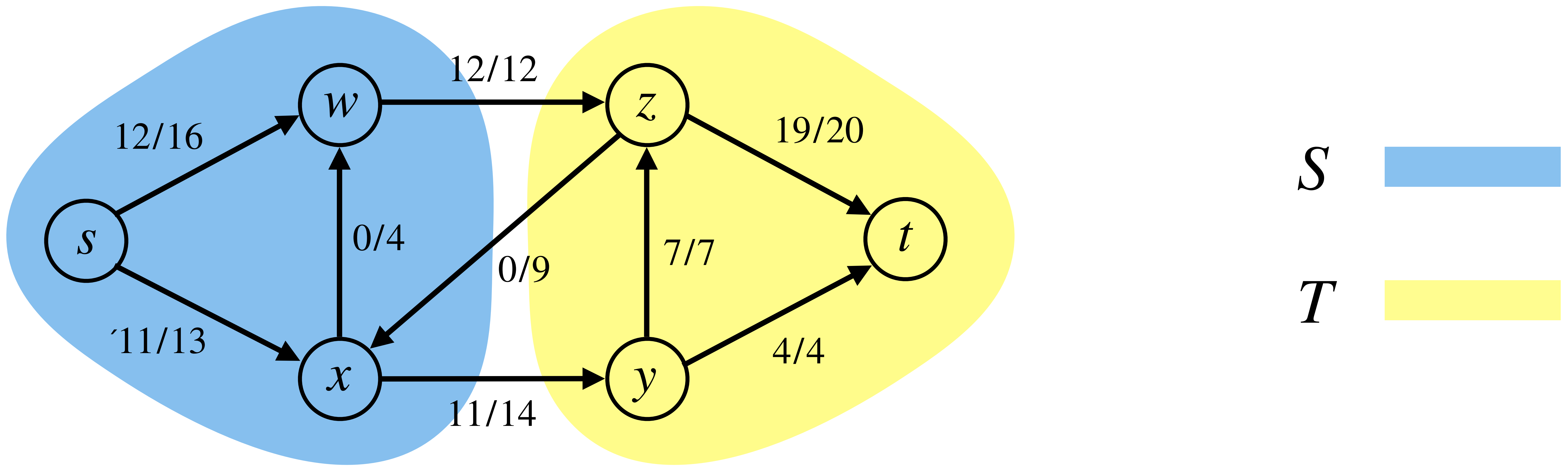

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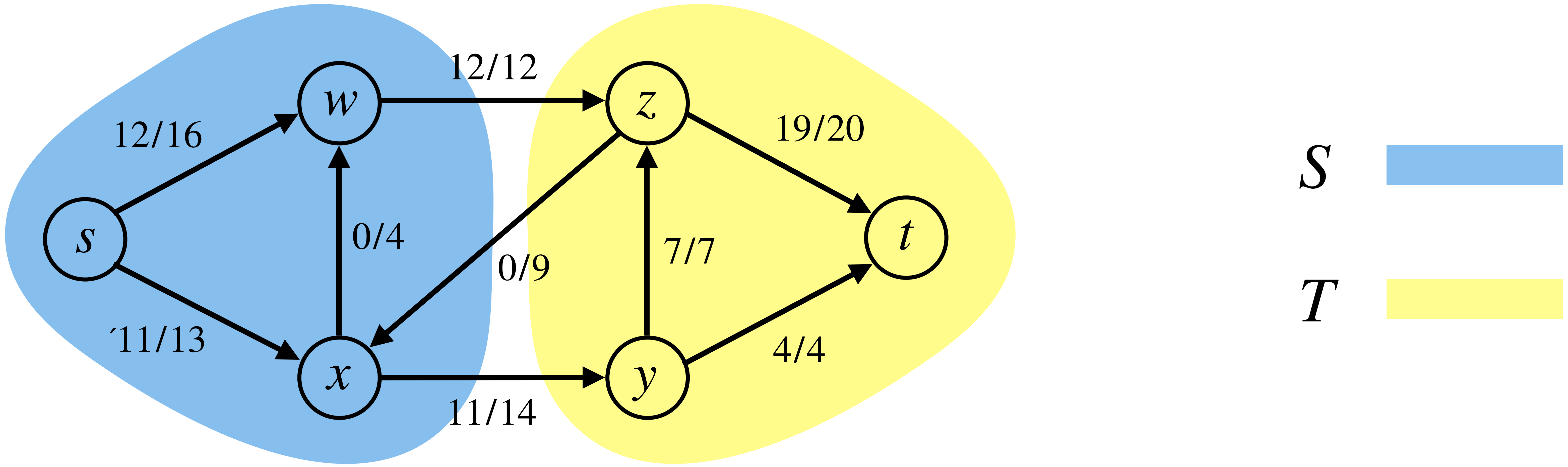
Examples:

$f(S,T) = 23$



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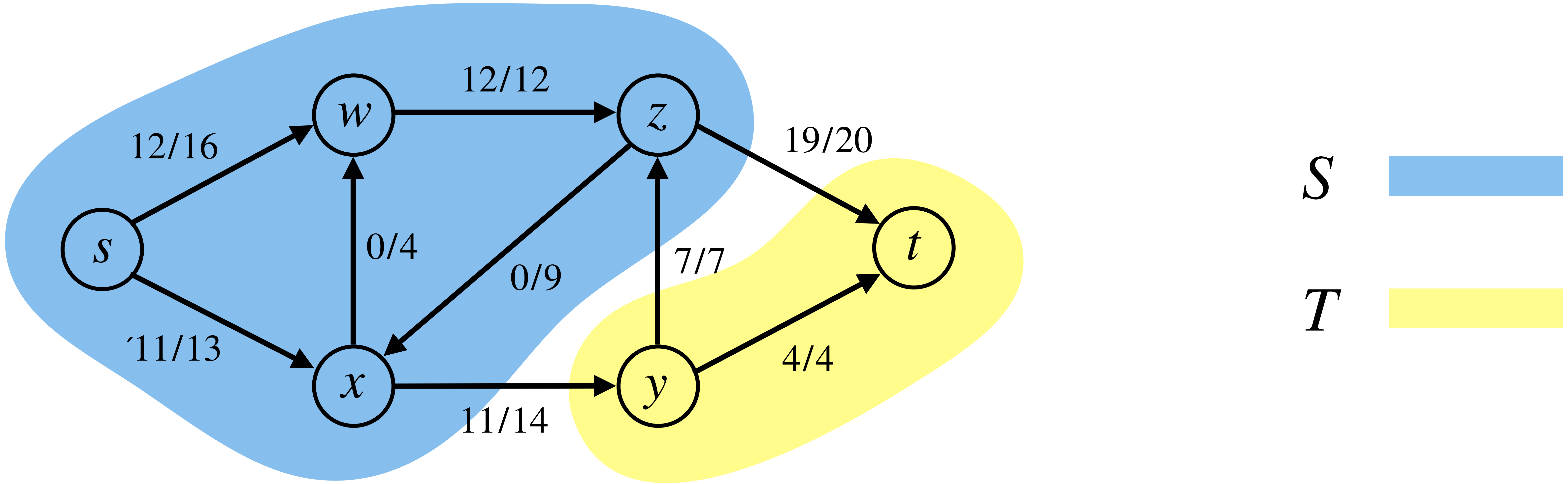

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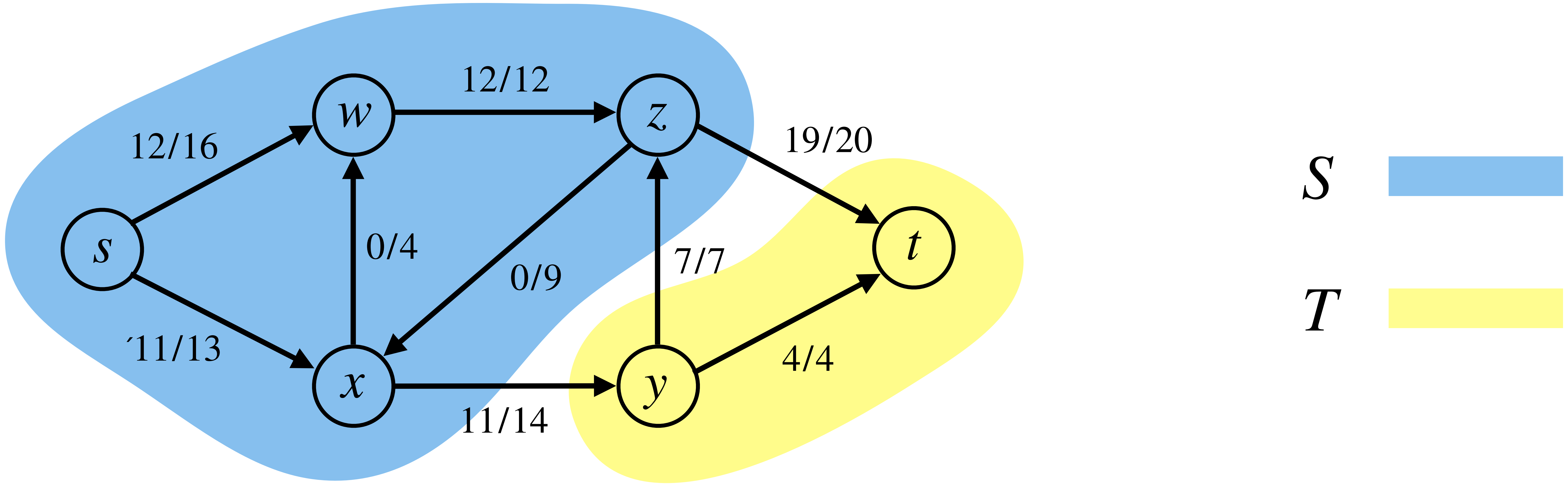

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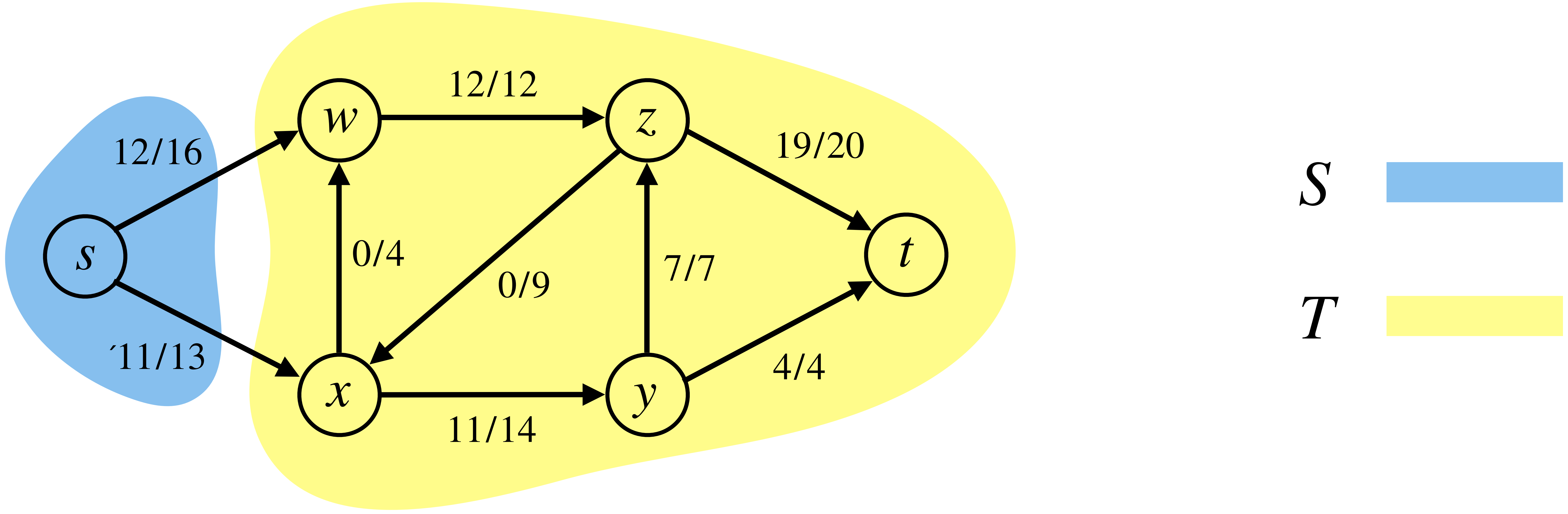

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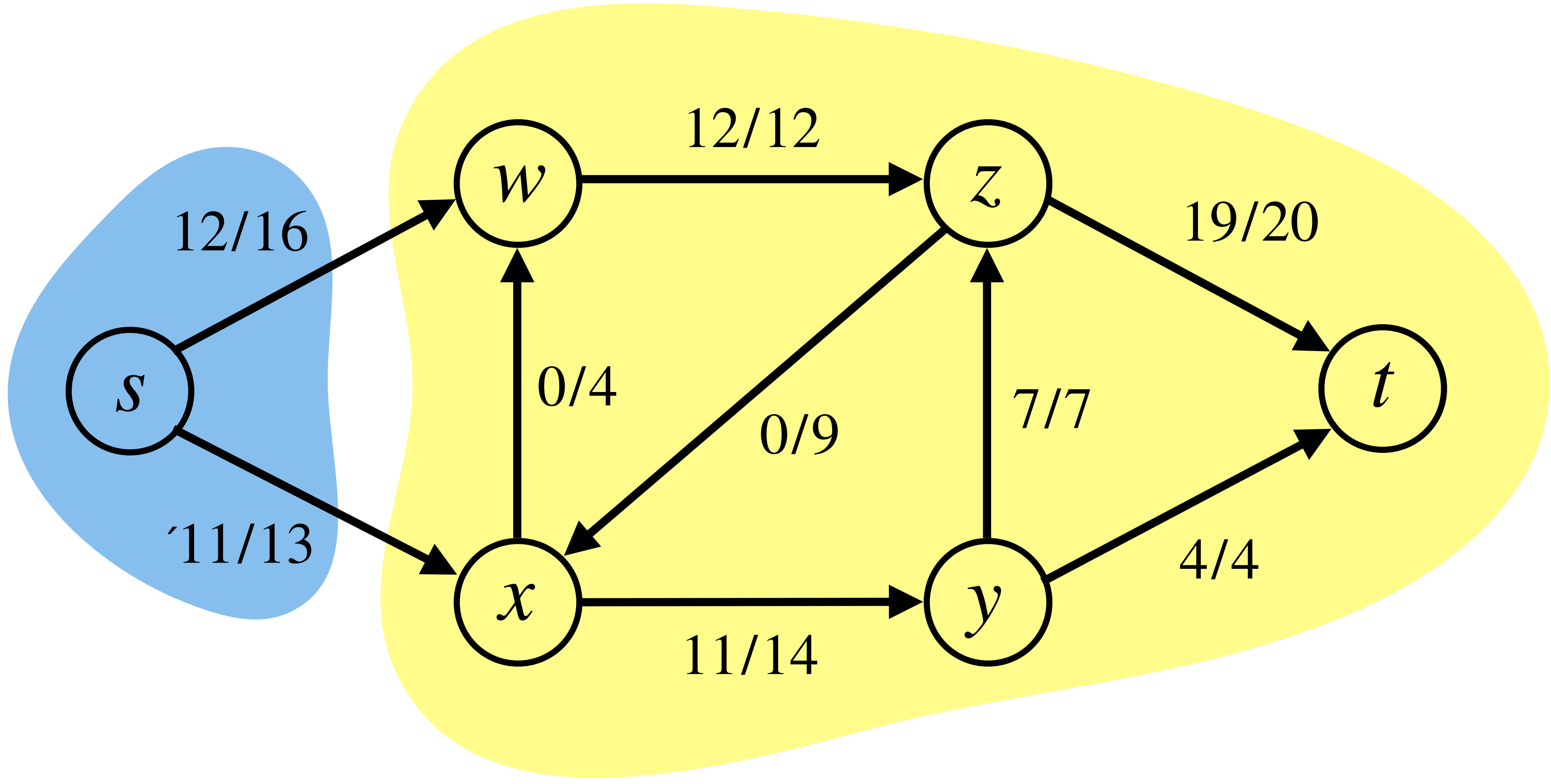

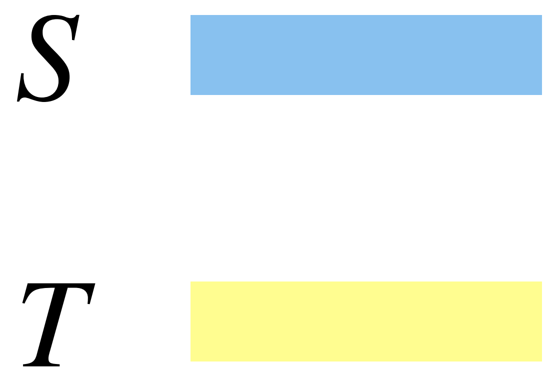

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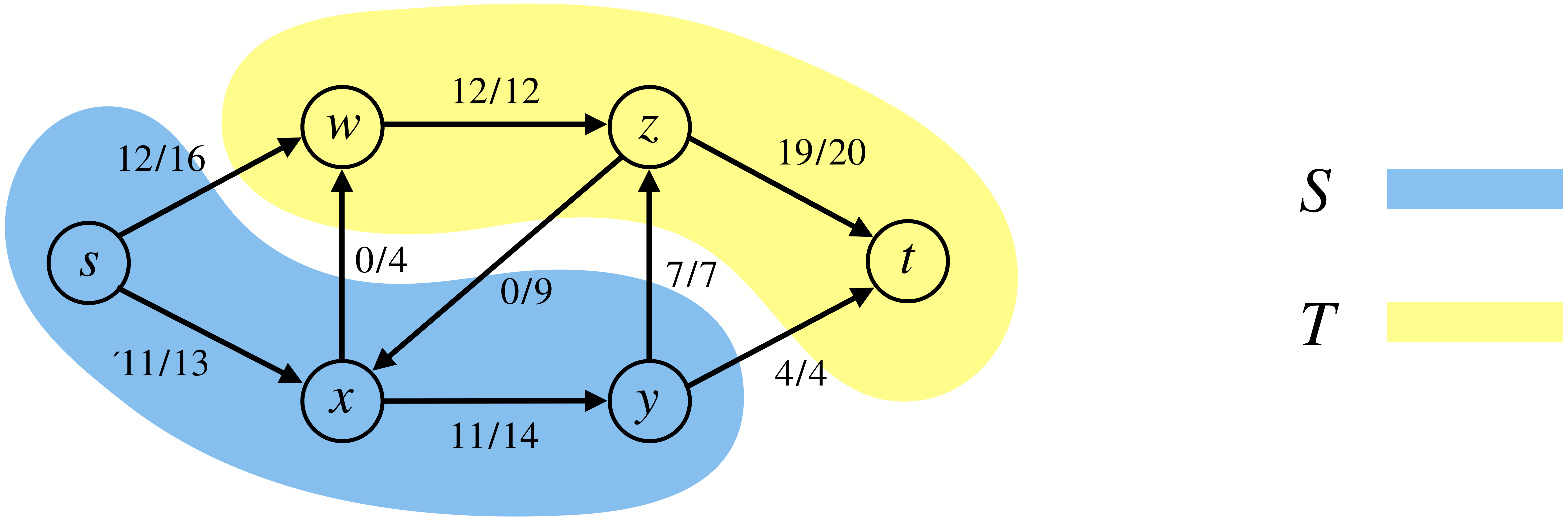

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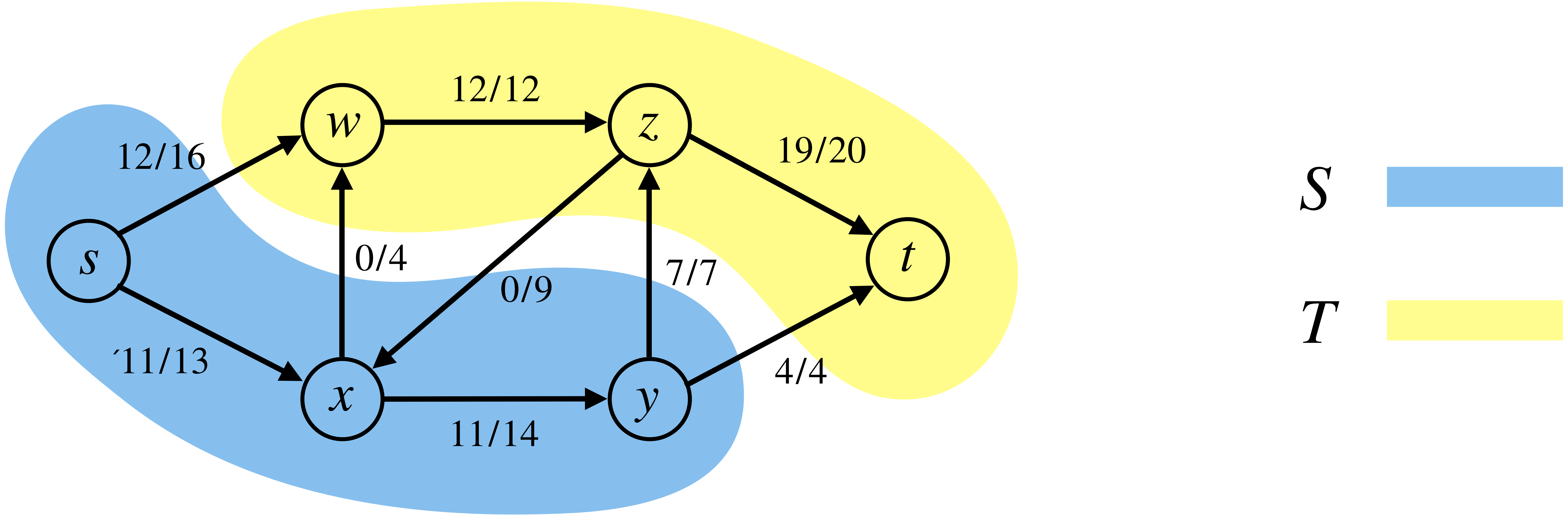

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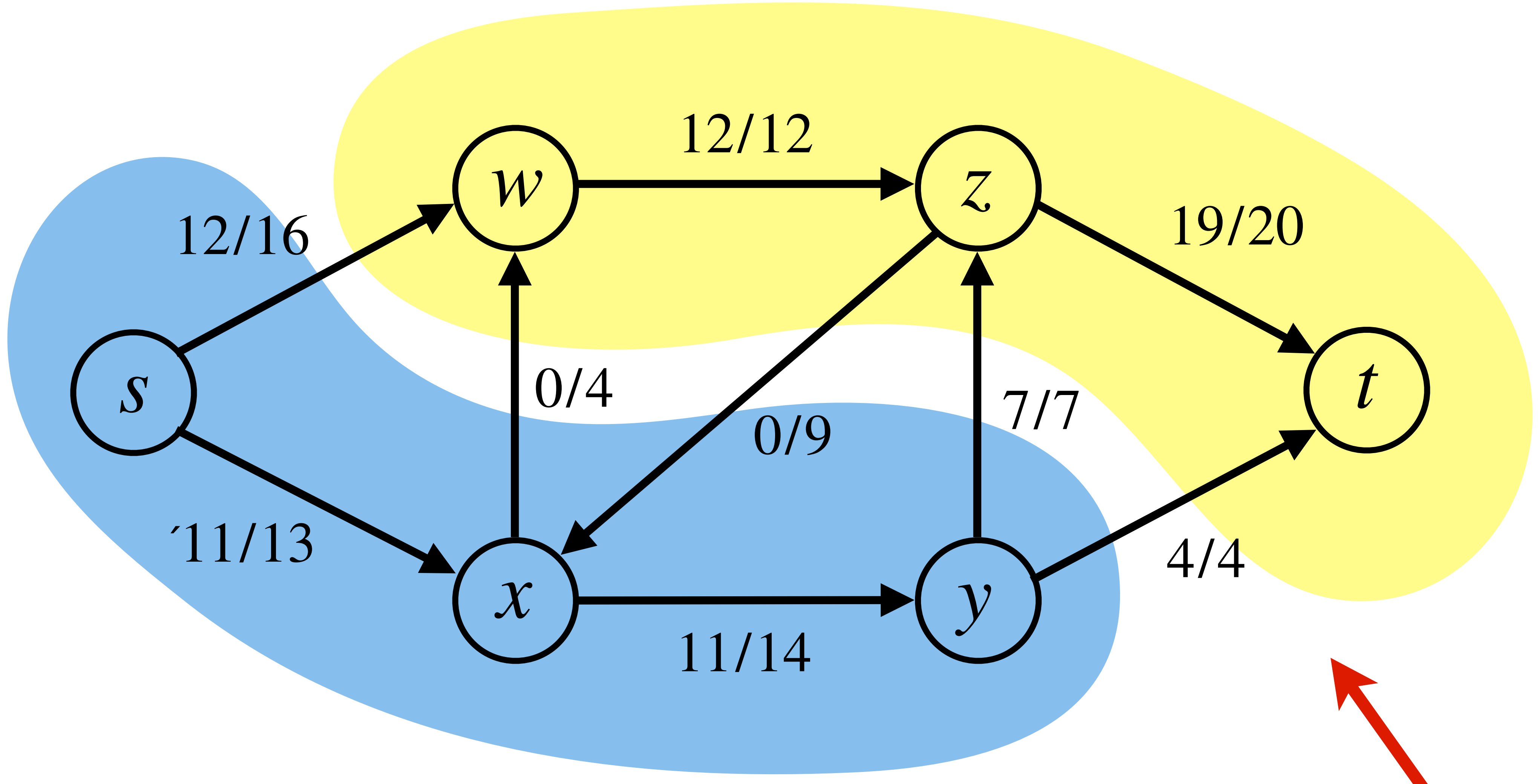

S

T

Why the net flow is the same for every cut?